Semitransitive and transitive subsemigroups of the inverse symmetric semigroups *

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Abstract

We classify minimal transitive subsemigroups of the finitary inverse symmetric semigroup modulo the classification of minimal transitive subgroups of finite symmetric groups; and semitransitive subsemigroups of the finite inverse symmetric semigroup of the minimal cardinality modulo the classification of transitive subgroups of the minimal cardinality of finite symmetric groups.

1 Introduction

An action of a semigroup S on a set X is said to be *transitive* if for every odered pair (x, y) in $X \times X$ there is an element of S that maps x to y. Recently, a weaker notion of semitransitivity was introduced by Rosenthal and Troitsky [13]. An action of S on X is *semitransitive* if for every ordered pair (x, y) in $X \times X$ there is an element φ in S such that either $x = y\varphi$ or $y = x\varphi$. So far the research of semitransitivity was mostly focused on the linear case, where X is a vector space and S consists of linear maps. Both can have some additional structure. For example, Rosenthal and Troitsky considered subalgebras of bounded linear operators on a Banach space. Semitransitive actions on a vector space of algebras and semigroups of linear operators were considered in [3], of vector spaces of linear operators in [1, 12, 15] and of Jordan algebras in [2]. Among other things, minimal semitransitive

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algebras were characterized in [13] as those simultaneously similar to the algebra of upper triangular Toeplitz operators (that is, the algebra generated by the identity and a nilpotent matrix of index n). Semitransitive actions of vector spaces S of linear operators on a vector space X were studied also in [12], where k-fold semitransitivity was also considered and Jacobson's density theorem for rings was extended. Examples of minimal semitransitive subspaces of linear maps acting on a finite-dimensional vector space that have no trivial invariant subspaces are given in [15] and in [1] it is proved that if the dimensions of a vector spaces S and X are equal then S is triangularizable.

In this paper, we study semigroups of partial transformations of a set X (see [4]). A semigroup of partial transformations of a set X is called *transitive* provided that for every ordered pair $(x, y) \in X \times X$ there is some $\varphi \in S$ such that $x\varphi = y$, and it is *semitransitive* provided that for every $(x, y) \in X \times X$ there is some $\varphi \in S$ such that either $x\varphi = y$ or $y\varphi = x$. The definitions ensure that transitivity of S implies its semitransitivity. If S is an inverse semigroup (in particular, if it is a group) then the converse is also true. However, in general, there are semitransitive semigroups which are not transitive.

We note that Schein [14] proved a number of results on transitive effective representations of inverse semigroups by partial one-to-one transformations of sets. Though to the best of our knowledge transitive and semitransitive semigroups of partial one-to-one transformations themselves have not been dealt with in the literature.

The aim of our paper is to give a classification of minimal transitive subsemigroups of the finitary inverse symmetric semigroup modulo the classification of minimal transitive subgroups of finite symmetric groups and the classification of semitransitive subsemigroups of the finite inverse symmetric semigroup of the minimal cardinality modulo the classification of transitive subgroups of the minimal cardinality of finite symmetric groups. It is well known that the classification of minimal transitive subgroups of finite symmetric groups is a difficult task. Namely, suppose that a finite group G acts faithfully and transitively on a finite set X and write $|X| = n$ for the cardinality of X. We view G as a subgroup of the symmetric group S_n of degree n. If $X = G$ and the action of G is by multiplication then we say that G acts regularly. Note that, in general, the transitivity implies that $|G| \geq |X|$. Igo Dak Tai [17] showed that if $n \neq p, p^2$, where p is a prime, then there exists a minimal transitive subgroup G of S_n that is not regular and such that $|G| > n$. Suprunenko [16] showed that if G is a minimal solvable transitive subgroup of S_n, $n = pq$, p and q distinct primes such that $p > q$ and $q \nmid (p-1)$, then G is either cyclic of order pq or a minimal nonabelian group of order $p^m q$ or pq^l, where m is the order of p in the multiplicative group of the Galois field $GF(q)$ and l the multiplicative order of q in $GF(p)$. The remaining case $q|(p-1)$ was treated by Kopylova [8]. Kopylova in [9] studied the structure of those subgroups that occur as minimal (nonregular) transitive subgroups of S_n. She showed that A_5 occurs as a minimal transitive subgroup of S_{10}. More recently, Hulpke [7] (see also his PhD thesis [6]) listed all the transitive (nonregular) groups up to degree $n = 30$.

We conclude the introduction with a brief description of the setup of the paper. In the second section we recall the definitions needed in the sequel and give the classification of minimal transitive subsemigroups of the finitary inverse symmetric semigroup modulo the classification of minimal transitive subgroups of finite symmetric groups. In the third section we classify semitransitive subsemigroups of the finite inverse symmetric semigroup of the minimal cardinality modulo the classification of transitive subgroups of the minimal cardinality of finite symmetric groups.

2 Classification of minimal transitive subsemigroups

We start by recalling standard definitions and elementary properties of regular and inverse semigroups which we use in the sequel [5, 10, 11]. Let S be a semigroup. Two elements $a, b \in S$ are called *mutually inverse* provided that $aba = a$ and $bab = b$. Whenever the stated equalities hold, we also say that a is an inverse of b, and b is an inverse of a. An element $a \in S$ is called *regular* provided that it possesses at least one inverse element. For $a \in S$ to be regular it is enough to require that there is $b \in S$ such that $aba = a$ (then a and bab are mutually inverse). The semigroup S is called *regular* provided that every $a \in S$ is regular. Further, S is called *inverse* provided that every element in S possesses a unique inverse. Equivalently, S is inverse if and only if it is regular and its idempotents commute. Therefore, a regular subsemigroup of an inverse semigroup is necessarily inverse.

By $\mathcal{IS}(X)$ we denote the *full inverse symmetric semigroup* over an underlying set X. (See [5, 10, 11].) In the case when X is a finite set and $n = |X|$ we write $\mathcal{IS}_n$ for $\mathcal{IS}(X)$ and take the convention that $X = \{1, \dots, n\}$. In the case when X is infinite we assume that $X \supseteq \mathbb{N}$. If X is infinite then all elements of $\mathcal{IS}(X)$ of finite ranks form a subsemigroup (and even an ideal). We denote this semigroup by $\mathcal{IS}_{fin}(X)$ and refer to it as to *the semigroup of all finitary partial permutations of* X. Accordingly, we call an element of finite rank a *finitary element*. We note that when X is a finite set, one obviously has $\mathcal{IS}_n = \mathcal{IS}_{fin}(X)$.

From now on, by the inverse element to a given $a \in \mathcal{IS}(X)$ we mean the element a^{-1} which is inverse to a in $\mathcal{IS}(X)$.

Proposition 2.1. *Let S be a transitive subsemigroup of $\mathcal{IS}(X)$ and I a non-zero ideal of S. Then I is a transitive ideal of $\mathcal{IS}(X)$.*

Proof. Let $\varphi \in I$, $\varphi \neq 0$. Consider any $i \in \mathrm{dom}\varphi$. Suppose $j = i\varphi$. Let $k, l \in X$. Transitivity of S ensures that there exist some $\psi_1, \psi_2 \in S$ satisfying $k\psi_1 = i$ and $j\psi_2 = l$. Then $k\psi_1\varphi\psi_2 = l$, and thus $\psi_1\varphi\psi_2 \in I$. $\square$

Recall that a semigroup S is called *simple* if it does not possess proper ideals, and 0-*simple* if it has a zero element 0, $S^2 \neq \{0\}$ and S does not possess proper non-zero ideals.

Corollary 2.2. *A minimal transitive subsemigroup S of $\mathcal{IS}(X)$ is either 0-simple or simple depending on whether S contains the zero element or not.*

Proposition 2.3. *Suppose S is a transitive subsemigroup of $\mathcal{IS}(X)$ such that $S \cap \mathcal{IS}_{fin}(X)$ contains a non-zero element. Then there is an inverse transitive subsemigroup of $\mathcal{IS}(X)$ which is contained in $S \cap \mathcal{IS}_{fin}(X)$.*

Proof. As $S \cap \mathcal{IS}_{fin}(X)$ is a non-zero ideal of S then by Proposition 2.1 we have that $S \cap \mathcal{IS}_{fin}(X)$ is also a transitive subsemigroup of $\mathcal{IS}(X)$. Hence we can assume that all elements of S are finitary.

It is enough to show that S contains a regular transitive subsemigroup.

Let $i, j \in X$. First let us show that S contains a pair of elements $a_{i,j}$, $a_{i,j}^{-1}$ such that $a_{i,j}$ is mapping i to j. Consider an arbitrary element $\varphi \in S$ such that $i\varphi = j$. Consider any $\psi \in S$ such that $j\psi = i$ (the existence of such an element ψ is provided by the transitivity of S). Obviously, we have $\mathrm{rank}\varphi\psi\varphi \leq \mathrm{rank}\varphi$ and $\mathrm{rank}\psi\varphi\psi \leq \mathrm{rank}\psi$. Consider two possible cases.

Case 1. Suppose $\mathrm{rank}\varphi\psi\varphi = \mathrm{rank}\varphi$ or $\mathrm{rank}\psi\varphi\psi = \mathrm{rank}\psi$. Suppose that the first of these equalities holds (if this is not the case then we just switch φ and ψ). Set $\overline{\psi} = \psi\varphi\psi$. Then $\mathrm{rank}\overline{\psi} \leq \mathrm{rank}\psi$ and $j\overline{\psi} = i$. Observe that

$$\mathrm{rank}\overline{\psi}\varphi\overline{\psi} = \mathrm{rank}\psi\varphi\psi\varphi\psi = \mathrm{rank}\psi\varphi\psi = \mathrm{rank}\overline{\psi}$$

and similarly

$$\mathrm{rank}\varphi\overline{\psi}\varphi = \mathrm{rank}\varphi\psi\varphi\psi\varphi = \mathrm{rank}\varphi\psi\varphi = \mathrm{rank}\varphi.$$

Thus we replace ψ by $\overline{\psi}$ if needed and assume that $\mathrm{rank}\psi \leq \mathrm{rank}\varphi$. This together with $\mathrm{rank}\varphi\psi\varphi = \mathrm{rank}\varphi$ implies that $\mathrm{ran}\varphi = \mathrm{dom}\psi$ and $\mathrm{ran}\psi = \mathrm{dom}\varphi$. If $\varphi\psi\varphi = \varphi$, then $\psi = \varphi^{-1}$, and we are done. Otherwise, set $M = \mathrm{dom}\varphi$ and note that $\mathrm{dom}\varphi\psi = \mathrm{ran}\varphi\psi = M$. This implies that there is some

$k \geq 1$ such that $(\varphi\psi)^k$ equals the identity transformation of M. Fix such a k. Then we can write

$$\varphi = (\varphi\psi)^k\varphi = \varphi \cdot (\psi\varphi)^{k-1}\psi \cdot \varphi,$$

ensuring that $\alpha\varphi\alpha = \varphi^{-1}$, where $\alpha = (\psi\varphi)^{k-1}\psi$.

Case 2. Suppose $\text{rank}\varphi\psi\varphi < \text{rank}\varphi$ and $\text{rank}\psi\varphi\psi < \text{rank}\psi$. Set $\varphi_1 = \varphi\psi\varphi$ and $\psi_1 = \psi\varphi\psi$. Then $\text{rank}\varphi_1 < \text{rank}\varphi$ and $\text{rank}\psi_1 < \text{rank}\psi$. Note that we still have $i\varphi_1 = j$ and $j\psi_1 = i$. In particular, if $\text{rank}\varphi = 1$ or $\text{rank}\psi = 1$, this case does not occur.

Applying the argument above at most $min\{\text{rank}\varphi, \text{rank}\psi\} - 1$ times we either find an element mapping i to j, which, together with its inverse, lies in S, or find an element of rank 1 mapping i to j. But for such an element its inverse must also belong to S since for such an element Case 2 is not possible.

For every i, j in X, we fix some $a_{i,j}$ such that $a_{i,j}, a_{i,j}^{-1} \in S$ and $a_{i,j}$ maps i to j. We consider the subsemigroup T of S generated by all such pairs $a_{i,j}$ and $a_{i,j}^{-1}$. It is obviously transitive. Besides, it is regular since for any $b = b_1 \cdots b_m \in T$, where $b_1, \dots, b_m$ are generators of T, we know that the element $b' = b_m^{-1} \cdots b_1^{-1}$, which also belongs to T, is the inverse of b in $\mathcal{IS}(X)$. Thus b and b' form a pair of mutually inverse elements in T. It follows that T is a regular semigroup and the proof is complete. □

Remark 2.4. *In Proposition 2.3 the requirement that $S \cap \mathcal{IS}_{fin}(X)$ contains a non-zero element S is essential. If this requirement is not satisfied, S may not contain an inverse transitive subsemigroup as the following example illustrates.*

Example 2.5. Let X be infinite. Let S be a subsemigroup of $\mathcal{IS}(X)$ consisting of all $\varphi \in \mathcal{IS}(X)$ such that $\text{dom}\varphi = X$ and $\text{ran}\varphi \neq X$. Obviously, S is transitive. Besides, S is idempotent-free. It follows that S does not contain inverse subsemigroups.

In order to formulate and prove the classification of minimal transitive subsemigroups of $\mathcal{IS}_{fin}(X)$ we recall the definition of a Brandt semigroup and then establish how Brandt subsemigroups of $\mathcal{IS}_{fin}(X)$ are built.

Let G be a group, $0 \notin G$ some symbol, which we call zero, and I a set. Let $B(G, I)$ be the set of all matrices with entries from $G \bigcup \{0\}$, whose rows and columns are indexed by I, and which contain at most one non-zero element. We denote the zero matrix just by 0, and the matrix whose the only entry from G equals g and is in the position (i, j) by $M(i, g, j)$. The product in $B(G, I)$ is given by

$$M(i,g,j)M(k,h,l) = \begin{cases} M(i,gh,l), & \text{if} j = k; \\ 0, & \text{otherwise}, \end{cases} \tag{1}$$

and the product with 0 at either side is again 0. The semigroup $B(G, I)$ is called a *Brandt semigroup* (see [5, 10, 11]). Brandt semigroups are inverse completely 0-simple semigroups, and every completely 0-simple inverse semigroup is isomorphic to some Brandt semigroup.

Corollary 2.6. *A subsemigroup S of $\mathcal{IS}_{fin}(X)$ is a minimal transitive subsemigroup of $\mathcal{IS}_{fin}(X)$, which is not a group, if and only if S is a minimal transitive Brandt subsemigroup of $\mathcal{IS}_{fin}(X)$.*

Proof. The statement follows from Corollary 2.2, Proposition 2.3 and the fact that a simple inverse semigroup is actually a group. □

Now we determine the structure of transitive Brandt subsemigroups of $\mathcal{IS}_{fin}(X)$. Let I be an index set, $|I| > 1$. Let, further, $\{M_i,\ i \in I\}$ be a collection of pairwise disjoint subsets of X such that $X = \cup_{i\in I} M_i$ and all M_i-s are of the same finite cardinality (this, in particular, implies that X and I must have the same cardinality whenever X is infinite). Suppose $1 \in I$ and let G be a subgroup of $\mathcal{S}(M_1)$. For every $i \in I$ fix some bijection π_i from M_1 to M_i. For every $i, j \in I$ and $g \in G$ set $M(i, g, j) = \pi_i^{-1} g \pi_j$.

Proposition 2.7. *1. All $M(i, g, j)$ together with 0 form a Brandt subsemigroup of $\mathcal{IS}_{fin}(X)$, which we denote by $B(G; M_i; \pi_i)_{i\in I}$.*

2. $B(G; M_i; \pi_i)_{i\in I} \subseteq B(H; M_i; \pi_i)_{i\in I}$ if and only if $G < H$.

3. $B(G; M_i; \pi_i)_{i\in I}$ is a transitive subsemigroup of $\mathcal{IS}_{fin}(X)$ if and only if G is a transitive subgroup of $\mathcal{S}(M_1)$.

4. Every transitive Brandt subsemigroup of $\mathcal{IS}_{fin}(X)$ coincides with some $B(G; M_i; \pi_i)_{i\in I}$, where G is a transitive subgroup of $\mathcal{S}(M_1)$.

Proof. It follows from the definition of $B(G; M_i; \pi_i)_{i\in I}$ that the multiplication in it obeys the rule (1), and therefore 1 is proven.

The second claim is straightforward.

Suppose G is a transitive subgroup of $\mathcal{S}(M_1)$ and take arbitrary $k, l \in X$. Suppose $k \in M_i$, $l \in M_j$. We can map k and l to M_1 by $\varphi = M(i, e, 1)$ and $\psi = M(j, e, 1)$ respectively. Further, $k\varphi$ can be mapped to $l\psi$ by some $\tau = M(1, g, 1)$ by the transitivity of G. It follows that $\varphi\tau\psi^{-1}$ maps k to l, implying the transitivity of $B(G; M_i; \pi_i)_{i\in I}$. Further, the mappings from M_1 to itself are given by the elements of the form $M(1, g, 1)$ where $g \in G$. Thus, if G is not transitive not every element in M_1 can be mapped to any element of M_1 by $B(G; M_i; \pi_i)_{i\in I}$. Hence, 3 is proven.

Now, suppose that S is a transitive Brandt subsemigroup of $\mathcal{IS}_{fin}(X)$. Consider any element $\varphi \in S$. Suppose $k = \text{rank}\varphi$. Then all the non-zero elements in S must also have rank k (since otherwise all elements of the smallest positive rank in S would form a transitive non-zero ideal). Denote $M_1 = \mathrm{dom}\varphi$ and fix $1 \in M_1$. We note that for every element in S its domain and range should either coincide with M_1 or be disjoint with it, as otherwise we would be able to find elements with positive ranks strictly less than k. Let $i \in X$. There is π in S mapping 1 to i. Since $\mathrm{dom}\pi \cap M_1 \neq \varnothing$ we have that $\mathrm{dom}\pi = M_1$. Similarly we conclude that $\mathrm{ran}\pi = M_1$ if $i \in M_1$, and $\mathrm{ran}\pi \cap M_1 = \varnothing$ if $i \notin M_1$. This argument ensures that there is a decomposition $X = \cup_{i\in I} M_i$ such that M_i-s are pairwise disjoint and all M_i-s are of the same finite cardinality. And, moreover, for every $i \in I$ there is some $\pi_i \in S$ with $\mathrm{dom}\pi_i = M_1$ and $\mathrm{ran}\pi_i = M_i$.

To map an element of M_1 to another element of M_1 we have to act by some π with $\mathrm{dom}\pi = \mathrm{ran}\pi = M_1$, that is, by an element of a maximal subgroup $\mathcal{S}(M_1)$. It follows that $S \cap \mathcal{S}(M_1)$ is a transitive subgroup of $\mathcal{S}(M_1)$ which we denote by G.

We know that S contains G, and all π_i, $i \in I$. Consider the semigroup S', generated as an inverse semigroup by G and π_i, $i \in I$. It equals $B(G; M_i; \pi_i)_{i\in I}$ and thus, in particular, is transitive. We are left to conclude that it must coincide with S in view of the minimality of S. Therefore, 4 is also proven. $\square$

We note that $B = B(G; M_i; \pi_i)_{i\in I}$ and $B' = B(G; M_i; \pi'_i)_{i\in I}$ are isomorphic. To see this, for each $i \in I$ we fix $g_i \in \mathcal{S}(M_1)$ such that $\pi'_i = \pi_i g_i$, and make sure that the map sending $M(i, g, j)$ in B to $g_i^{-1} M(i, g, j) g_j$ in B' is an isomorphism. Further,

$$B(G; M_i; \pi_i)_{i\in I} \simeq B(\pi_1^{-1} G \pi_1; M_i; \pi_1^{-1}\pi_i)_{i\in I}.$$

Therefore, we can always choose S such that $\pi_1 = e$. We also note that $B = B'$ if and only if all $\pi_i \pi_j'^{-1}$-s lie in the normalizer of G in $\mathcal{S}(M_1)$.

In the following theorem we give a classification of minimal transitive subsemigroups of $\mathcal{IS}_{fin}(X)$ modulo the classification of minimal transitive subgroups of the finite full symmetric groups $\mathcal{S}_k$.

Theorem 2.8.
1. *Let X be an infinite set. Then the semigroups $B(G; M_i; \pi_i)_{i\in I}$, where G is a minimal transitive subgroup of $\mathcal{S}(M_1)$, constitute the full list of minimal transitive subsemigroups of $\mathcal{IS}_{fin}(X)$.*

2. *Let X be a finite set, and $|X| = n$. Then the semigroups $B(G; M_i; \pi_i)_{i\in I}$, where $|I| > 1$ and G is a minimal transitive subgroup of $\mathcal{S}(M_1)$; and all minimal transitive subgroups of $\mathcal{S}(X)$, constitute the full list of minimal transitive subsemigroups of $\mathcal{IS}(X)$.*

Proof. Suppose first that X is infinite. Then $\mathcal{IS}_{fin}(X)$ does not possess transitive subsemigroups which are groups. To see this, we note that any subgroup of $\mathcal{IS}_{fin}(X)$ is a subgroup of $\mathcal{S}(M)$, where M is a finite subset of X and therefore such a subgroup acts on M only, which means that it can not act transitively on X. Now the first claim follows from Corollary 2.6 and Items 2 and 4 of Proposition 2.7.

To prove the second claim we apply similar arguments. The only difference with the previous case is that for finite X there are transitive subsemigroups which are groups - those are transitive subgroups of $S(X)$. $\square$

Let us look in more detail at minimal transitive subsemigroups of $\mathcal{IS}_n$, which are not groups. Consider such a subsemigroup $B(G; M_i; \pi_i)_{i\in I}$. Since X is finite, so is also I. Let $|I| = k$ and suppose $I = \{1, \dots, k\}$. We know that all M_i-s have the same cardinality, therefore $|M_i| = \frac{n}{k}$, $i \in I$ (in particular, n is divisible by k). Let us write $B(G; M_1, \dots, M_k; \pi_1, \dots \pi_k)$ for $B(G; M_i; \pi_i)_{i\in I}$.

Theorem 2.9. *Let k be a divisor of n.*

1. *All non-zero elements of $B(G; M_1, \dots, M_k; \pi_1, \dots \pi_k)$ have rank $\frac{n}{k}$.*

2. *The semigroup $B(G; M_1, \dots, M_k; \pi_1, \dots \pi_k)$ with non-zero elements of rank $\frac{n}{k} < n$ has cardinality $k^2x + 1$, where $x = |G|$. It is a minimal transitive subsemigroup of $\mathcal{IS}_n$.*

3. *Let $t(k)$ be the number of minimal transitive subgroups of $\mathcal{S}_k$ up to isomorphism. Then, up to isomorphism, the number of minimal transitive subsemigroups of $\mathcal{IS}_n$ equals*

$$\sum_{k \text{ divides } n} t(k).$$

Proof. By Theorem 2.8, Item 2 we know that $B(G; M_1, \dots, M_k; \pi_1, \dots \pi_k)$ has elements of rank $\frac{n}{k}$. But all non-zero elements of $B(G; M_1, \dots, M_k; \pi_1, \dots \pi_k)$ are of the same rank (see the third paragraph of the proof of Proposition 2.7), which proves 1.

Brandt semigroups with isomorphic groups and equicardinal index sets are isomorphic. Besides, the cardinality of a Brandt semigroup with the index set I and group G is given by $|I|^2|G| + 1$, which implies 2.

The statement 3 is obvious. $\square$

We conclude this section by an example of a minimal transitive subsemigroup of $\mathcal{IS}_8$.

Example 2.10. Let $k = 2$. Choose a partition of $\{1, \ldots, 8\}$ into two 4-element subsets, for example, $\{1, 2, 3, 4\}$, $\{5, 6, 7, 8\}$. Denote the first set by M_1, and the second one by M_2. Now we choose a cyclic subgroup of order 4 in $\mathcal{S}(M_1)$, for example, we take $G = \langle(1, 2, 3, 4)\rangle$. Choose a bijection $\pi_2 : M_1 \to M_2$, for example, $\pi_2(i) = i + 4$, $i = 1, 2, 3, 4$. We obtain the semigroup $B(G; M_1, M_2; e, \pi_2)$, which has $4 \cdot 2^2 + 1 = 17$ elements which are listed below using the cycle-chain notation (see [4] and the references therein):

$$\begin{array}{ll}
(1,2,3,4)5]6]7]8]; & (1,3)(2,4)5]6]7]8]; \\
(1,4,3,2)5]6]7]8]; & (1)(2)(3)(4)5]6]7]8]; \\
(1,5](2,6](3,7](4,8]; & (1,6](2,7](3,8](4,5]; \\
(1,7](2,8](3,5](4,6]; & (1,8](2,5](3,6](4,7]; \\
(5,1](6,2](7,3](8,4]; & (6,1](7,2](8,3](5,4]; \\
(7,1](8,2](5,3](6,4]; & (8,1](5,2](6,3](7,4]; \\
(5,6,7,8)1]2]3]4]; & (5,7)(6,8)1]2]3]4]; \\
(5,8,7,6)1]2]3]4]; & (5)(6)(7)(8)1]2]3]4] \\
\text{and } 0. &
\end{array}$$

3 Classification of semitransitive subsemigroups of $\mathcal{IS}_n$ of the minimal cardinality

In this section we switch to semitransitive subsemigroups of $\mathcal{IS}_{fin}(X)$. As any transitive subsemigroup is automatically semitransitive, and transitive subsemigroups have been dealt with in the previous section, we can limit our attention to semitransitive subsemigroups of $\mathcal{IS}_{fin}(X)$, which are not transitive.

Let S be a semigroup with the zero element 0. A non-zero element $a \in S$ is called *nilpotent* provided that some power of a equals 0.

In what follows let 0 stand for the zero element of $\mathcal{IS}_{fin}(X)$, that is 0 is the nowhere defined partial permutation of X.

Proposition 3.1. *A semitransitive subsemigroup of $\mathcal{IS}_{fin}(X)$, that is not transitive, contains* 0 *and a nilpotent element.*

Proof. Let S be a semitransitive subsemigroup of $\mathcal{IS}_{fin}(X)$ that is not transitive. Then S contains an element of a positive rank strictly less than $|X|$. This is obvious for infinite X. Suppose X is finite, and S contains only elements of rank $|X|$ and possibly 0. Then S is a subsemigroup of $\mathcal{IS}(X)$ with possibly adjoint 0. But a subsemigroup of a finite group is in fact a group. Then $S = H$ or $S = H \cup \{0\}$ where H is a semitransitive subgroup of $\mathcal{IS}(X)$. This implies that S is transitive, which contradicts the assumption.

Suppose S does not contain 0. Let $\varphi \in S$ be a non-zero element and $\text{rank}\varphi < |X|$. Then some power of φ is a non-zero idempotent, which also belongs to S and has rank less or equal than that of φ. Let $\psi \in S$ be a non-zero idempotent with the minimum possible rank (among all elements of S). Let $Y = \text{dom}\psi = \text{ran}\psi$. Set $Z = X \setminus Y$. Since both Y and Z are non-empty, we can choose some $i \in Y$ and $j \in Z$. Let α be the element in S which maps either i to j or vice versa. Then $0 < \text{rank}(\psi\alpha\psi) < \text{rank}\psi$. The obtained contradiction shows, that S must contain 0.

Finally, we show that S contains a nilpotent element. Let ψ and α be as in the preceding paragraph. By construction, we have $Y \cap \text{dom}\alpha \neq \varnothing$ or $Y \cap \text{ran}\alpha \neq \varnothing$, implying that at least one of the elements $\alpha\psi$ or $\psi\alpha$ is non-zero. Suppose $\alpha\psi \neq 0$ (the other case is treated similarly). Since $\text{rank}(\psi\alpha\psi) < \text{rank}\psi$, it follows that $\psi\alpha\psi = 0$. Then $(\alpha\psi)^2 = \alpha(\psi\alpha\psi) = \alpha \cdot 0 = 0$, ensuring that $\alpha\psi$ is a nilpotent element. □

Let S be a subsemigroup of $\mathcal{IS}(X)$. An element $i \in X$ is called *cyclic with respect to* S provided that for every $j \in X$ there is $\varphi_j \in S$ such that $i\varphi_j = j$. In particular, if S is transitive then every $i \in X$ is cyclic.

We choose $i \in X$. Then iS^1 is the set $\{i\} \cup iS$, that is, the set of all those elements of X where i can be mapped by partial permutations from S and by the identity map. For $i, j \in X$ set $i \leq_r j$ if there is $\varphi \in S$ such that $i\varphi = j$. The latter is true if and only if the inclusion $iS^1 \supseteq jS^1$ holds. If S is a semitransitive subsemigroup then the relation $\leq_r$ is a linear preorder on X. This implies that the relation r on X defined via irj if and only if $i \leq_r j$ and $j \leq_r i$ is an equivalence relation. Moreover, the equivalence classes are naturally linearly ordered by the order induced by $\leq_r$ which we denote just by $\leq$. Suppose, X is finite and $M_1, \dots, M_k$ are all the equivalence classes with respect to r and $M_1 > M_2 > \cdots > M_k$. In particular, M_1 is the set of all elements that are cyclic with respect to S. The action of S on each M_i is transitive and for each pair i, j with $i < j$ there is no element in S that maps an element of M_j to an element of M_i. Therefore, for each $x \in M_i$ and $y \in M_j$ there is an element φ in S such that $y = x\varphi$ by semitransitivity.

Example 3.2. If X is infinite, there are semitransitive subsemigroups of $\mathcal{IS}_{fin}(X)$ without cyclic elements. For instance, take $X = \mathbb{Z}$. For every $i, j \in X$ with $i \leq j$ let $\varphi_{i,j}$ be the element of rank 1 which maps i to j. All the elements $\varphi_{i,j}$, together with 0, form a semitransitive subsemigroup of $\mathcal{IS}_{fin}(X)$. However, with respect to this semigroup there is no cyclic element in X.

By the reason above in what follows we restrict our attention to the case when the set X is finite. We assume that $X = \{1, 2 \dots, n\}$ and write $\mathcal{IS}_n$

for $\mathcal{IS}(X)$.

The discussion above leads to the following lower bound on the cardinality of a semitransitive subsemigroup of $\mathcal{IS}_n$.

Proposition 3.3. *The cardinality of a semitransitive but not transitive subsemigroup of $\mathcal{IS}_n$ is greater than or equal to $n+1$.*

Proof. Let S be a semitransitive (and not transitive) subsemigroup of $\mathcal{IS}_n$, and $i \in X$ an element that is cyclic with respect to S. Then S must contain elements that send i to $1, 2, \ldots, n$. Besides, S contains 0 by Proposition 3.1. It follows that the cardinality of S should be at least $n+1$. □

Now, our goal is to provide a construction which gives examples of semitransitive and not transitive subsemigroups of $\mathcal{IS}_n$ of cardinality $n+1$, and then to prove that every such a subsemigroup is equal to one already constructed.

Let $k > 1$ be a divisor of n. Consider a partition $X = M_1 \cup \cdots \cup M_k$, such that $M_i \cap M_j = \varnothing$ whenever $i \neq j$, and $|M_i| = \frac{n}{k}$ for every i. Suppose that $M_i = \{a_{i,1}, \ldots a_{i,\frac{n}{k}}\}$. Let G be some transitive permutation group acting on M_1. Fixing the bijection from M_1 to M_i, sending $a_{1,j}$ to $a_{i,j}$, $1 \leq j \leq \frac{n}{k}$, $2 \leq i \leq k$, one diagonally extends the action of G to the whole X. Consider the chain $(a_{1,1}, a_{2,1} \ldots, a_{k,1}] \in \mathcal{IS}(\{a_{1,1}, \ldots, a_{k,1}\})$. It generates a nilpotent semigroup T consisting of $k+1$ elements. We extend the action of T to X also diagonally using the bijections sending $a_{j,1}$ to $a_{j,i}$, $2 \leq i \leq k$, $1 \leq j \leq \frac{n}{k}$. Consider the subsemigroup S of $\mathcal{IS}_n$ generated by G and T^1. Obviously, S is semitransitive. Since the actions of G and T^1 on X commute, and no nonzero $s \in T$ acts in the same way as some gs_1, $g \in G, g \neq e$, $s_1 \in T^1$, it follows that S is equal to the Rees factor $(G \times T^1)/I$, where the ideal I consists of all elements $(g, 0)$, $g \in G$.

Example 3.4. Let $n = 8$ and $k = 4$. We choose $M_1 = \{1, 2\}$, $M_2 = \{3, 4\}$, $M_3 = \{5, 6\}$ and $M_4 = \{7, 8\}$. Suppose that $G = \{e, g\}$, where $g = (1,2)(3,4)(5,6)(7,8)$, and $T = \{\varphi, \varphi^2, \varphi^3, 0\}$, where $\varphi = (1,3,5,7](2,4,6,8]$. The semigroup $(G \times T^1)/I$ consists of the following elements:

$(e,1) = (1)(2)(3)(4)(5)(6)(7)(8)$; $(g,1) = (1,2)(3,4)(5,6)(7,8)$
$(e,\varphi) = (1,3,5,7](2,4,6,8]$; $(g,\varphi) = (1,4,5,8](2,3,6,7]$
$(e,\varphi^2) = (1,5](3,7](2,6](4,8]$; $(g,\varphi^2) = (1,6](2,5](3,8](4,7]$
$(e,\varphi^3) = (1,7](2,8]3]4]5]6]$: $(g,\varphi^3) = (1,8](2,7]3]4]5]6]$
and 0.

Theorem 3.5. *Let S be a semitransitive but not transitive subsemigroup of $\mathcal{IS}_n$ of cardinality $n+1$. Then the action of S on X coincides with the action of some $(G \times T^1)/I$ given in the construction above.*

We prove Theorem 3.5 in several steps. First of all, fix a semitransitive (but not transitive) subsemigroup of $\mathcal{IS}_n$. We consider the relation r on X, and the order $M_1 > \cdots > M_k$ on the r-classes. Note that $k > 1$ since S is not transitive. Denote the cardinality of M_i by m_i, $1 \le i \le k$.

Lemma 3.6. *For every non-zero $\varphi \in S$ we have the inclusions $M_1 \subseteq \mathrm{dom}(\varphi)$, $M_k \subseteq \mathrm{ran}(\varphi)$.*

Proof. Let $\varphi \in S$ and $i, j \in X$. We will say that φ *has an arrow* from i to j provided that $i \in \mathrm{dom}(\varphi)$ and $i\varphi = j$. Every element of M_1 is cyclic, thus S contains at least n arrows from i in total for every $i \in M_1$. Therefore there are at least $m_1 \cdot n$ arrows from the elements of M_1 if we run through all elements of S. But an element in S contains at most m_1 arrows from M_1, which implies that there are at least $\frac{m_1 \cdot n}{m_1} = n$ elements in S which have some arrows from M_1. Since S has precisely n non-zero elements (S does have the zero by Proposition 3.1), it follows that every non-zero element in S should have m_1 arrows from M_1. This means that $M_1 \subseteq \mathrm{dom}(\varphi)$ for every $\varphi \in S$. The second inclusion is established in the same fashion. □

Lemma 3.7. *Let $\varphi \in S$. Then either $M_1\varphi = M_1$ and $M_k\varphi = M_k$ or $M_1 \cap \mathrm{ran}(\varphi) = \varnothing$ and $M_k \cap \mathrm{dom}(\varphi) = \varnothing$. Moreover, $M_1\varphi = M_1$ holds if and only if $M_k\varphi = M_k$ holds, and $M_1 \cap \mathrm{ran}(\varphi) = \varnothing$ holds if and only if $M_k \cap \mathrm{dom}(\varphi) = \varnothing$ holds.*

Proof. Suppose that $M_1\varphi \ne M_1$ and that $M_1 \cap \mathrm{ran}(\varphi) \ne \varnothing$. Then there are $i, j \in M_1$ such that φ has an arrow from i to j. If $i \in \mathrm{dom}(\varphi^t)$ for all t then there is an h in M_1 such that $f = h\varphi \notin M_1$. Then the cycle-chain decomposition of φ has a cycle $(i, j, \dots)$ and a chain $(\dots, h, f, \dots]$. Otherwise, the cycle-chain decomposition of φ has a chain $(i, j, \dots, g, \dots]$, where $g \in M_s$ with $s > 1$. This means that there is some power φ^l such that either h or j does not belong to its domain while i still does. This contradicts Lemma 3.6. Similarly, one shows that if φ has an arrow from i to j with $i, j \in M_k$ then $M_k\varphi = M_k$. Finally, each of the cases $M_1\varphi = M_1$ and $M_k \cap \mathrm{dom}(\varphi) = \varnothing$; and $M_k\varphi = M_k$ and $M_1 \cap \mathrm{ran}(\varphi) = \varnothing$ is impossible, since otherwise there would exist a power φ^l such that it is non-zero with either M_k not in its range, or M_1 not in its domain, respectively, which again contradicts Lemma 3.6. □

Lemma 3.8. *There is an element $\varphi \in S$ with $\mathrm{dom}(\varphi) = M_1$ and $\mathrm{ran}(\varphi) = M_k$ (thus, in particular, $m_1 = m_k$).*

Proof. Let $\varphi \in S$. Suppose φ has an arrow from i to j for some $i \in M_1$ and $j \in M_k$. Then Lemma 3.7 implies that $M_1 \cap \mathrm{ran}(\varphi) = \varnothing$ and $M_k \cap \mathrm{dom}(\varphi) = \varnothing$. We will show that all arrows from elements of M_1 in φ go to elements of

M_k. Suppose that this is not the case and that φ has an arrow from x to y with $x \in M_1$ and $y \in M_l$ with $1 < l < k$. Consider some $\psi \in S$ such that $y \in \mathrm{dom}(\psi)$ and $y\psi = j$. It exists by semitransitivity and construction of the sets M_i. Since $M_k\psi \neq M_k$ we have $M_k \cap \mathrm{dom}(\varphi) = \varnothing$ by Lemma 3.7. Thus $j \notin \mathrm{dom}(\psi)$. Then $\varphi\psi \in S$ with $i \notin \mathrm{dom}(\varphi\psi)$ and $x \in \mathrm{dom}(\varphi\psi)$, which is impossible by Lemma 3.6. Similarly, one shows that all arrows to elements of M_k in φ go from elements of M_1.

Consider an arbitrary $\varphi \in S$. In view of the previous paragraph, we have only to consider the following two cases.

Case 1. Suppose all arrows from M_1 in φ go to M_k. In this case we show that in fact $\mathrm{dom}(\varphi) = M_1$, that is φ has no other arrows but those from M_1. If this were not the case, φ would have some arrow from $x \in M_s$ to $y \in M_t$ with $1 < s \leq t < k$. We choose $\psi \in S$ mapping y to M_k. It exists by semitransitivity. Invoking Lemma 3.7 we get $M_k \cap \mathrm{dom}(\psi) = \varnothing$ since $M_k\psi \neq M_k$. Now, $\varphi\psi$ is a non-zero element in S whose domain does not contain M_1, which is impossible by Lemma 3.6. This proves that φ has the property we are looking for.

Case 2. Suppose there is an arrow from M_1 in φ that does not go to M_k. Consider some arrow of φ, from i to j, with $i \in M_1$ and $j \in M_l$, $l < k$. There is $\psi \in S$ with an arrow from j to M_k. Then $\varphi\psi$ has an arrow from M_1 to M_k, which, in view of the first paragraph of this proof, implies that all arrows from M_1 in $\varphi\psi$ go to M_k. Now Case 1 ensures that $\varphi\psi$ is the required element. □

Proof of Theorem 3.5. We apply induction on the number of r-classes k. Consider first the case $k = 2$. From Lemma 3.8 we know that $m_1 = m_2$. Then, from Lemma 3.7 it follows that S has elements of two different types. The first type: the elements φ with $\mathrm{dom}(\varphi) = \mathrm{ran}(\varphi) = X$ such that $M_1\varphi = M_1$ and $M_2\varphi = M_2$. The second type: the elements φ with $\mathrm{dom}(\varphi) = M_1$ and $M_1\varphi = M_2$. Since elements of the first type act transitively on M_1, there are at least m_1 such elements. Fix some element φ of the second type. Multiplying it with different elements of the first type we obtain different elements of the second type, meaning that S has at least $2m_1 = n$ non-zero elements. It follows that the cardinality of the set of the elements of the first type is m_1, and the restrictions to M_1 of these elements form a transitive group of permutations of M_1. Let T be a semigroup generated by φ. We have that the action of S coincides with the action of $(G \times T^1)/I$, where the ideal I consists of all elements $(g, 0)$, $g \in G$, which finishes the proof of the induction base.

Let now $k \geq 3$. We construct a homomorphism γ from S to some semitransitive subsemigroup of $\mathcal{IS}(X \setminus M_k)$ with the r-classes $M_1 > \cdots > M_{k-1}$.

Let $\varphi \in S$. If $M_k\varphi = M_k$ then also $(X \setminus M_k)\varphi = X \setminus M_k$. In this case we set $\varphi\gamma$ to be equal to the restriction of φ to $X \setminus M_k$. Otherwise we have $M_k \cap \mathrm{dom}(\varphi) = \varnothing$ (by Lemma 3.7), meaning that in the cycle-chain notation each element of M_k occurs at the end of some non-empty chain. We define the element $\varphi\gamma$ by erasing the last elements of all chains in φ. This construction ensures that γ is homomorphic, and that $S\gamma$ is a semitransitive subsemigroup of $\mathcal{IS}(X \setminus M_k)$ with the r-classes $M_1 > \cdots > M_{k-1}$. Let us estimate its cardinality. By Lemma 3.8 we know that there is $\varphi \in S$ with $\mathrm{dom}(\varphi) = M_1$ and $\mathrm{ran}(\varphi) = M_k$. Besides, we have at least m_k elements ψ such that $M_k \subseteq \mathrm{dom}(\psi)$ (this follows from Lemma 3.7 and that a group acting transitively on a s-element set is at least of cardinality s). Therefore, considering all possible products $\varphi\psi$, we make sure that S has at least m_k different elements α with $\mathrm{dom}(\alpha) = M_1$ and $\mathrm{ran}(\alpha) = M_k$. All these elements, as well as the zero of S, are mapped by γ to the zero of $S\gamma$, implying that $S\gamma$ has at most $n + 1 - m_k = m_1 + \cdots + m_{k-1} + 1 = |X \setminus M_k| + 1$ elements. It follows that $S\gamma$ is a semitransitive subsemigroup of $\mathcal{IS}(X \setminus M_k)$ of the minimum cardinality, and the induction assumption can be implied. It follows that $m_1 = \cdots = m_k$, that the action of $S\gamma$ on $X \setminus M_k$ coincides with the action of some $(G \times T^1)/I$, where G is a m_1-element group acting transitively on M_1, T is generated by some nilpotent element φ all whose chains are of length $k - 1$ and go from M_1 through $M_2, \ldots$ to M_{k-1}. Let ψ be some γ-preimage of φ, and $\overline{T}$ the semigroup generated by it. Then S contains the subsemigroup $(G \times \overline{T}^1)/\overline{I}$ with the ideal $\overline{I}$ consisting of all elements $(g, 0)$, $g \in G$ (0 is the zero of S). Since this semigroup and S are of the same cardinality $n + 1$, we conclude that they must coincide. This completes the proof. □

The assumption of minimal cardinality in Theorem 3.5 implies that the cardinality of the sets M_i are all equal. If we only assume minimality then this is no longer the case as is shown by the following example.

Example 3.9. *Assume that* $X = \{1, 2, 3\}$ *and that* $M_1 = \{1, 2\}$ *and* $M_2 = \{3\}$. *Then* $S = \{(1, 2)(3), (1)(2)(3), (1, 3]2], (2, 3]1], 0\}$ *is a minimal semitransitive subsemigroup of* $\mathcal{IS}(X)$. *Its cardinality is equal to* 5, *which is greater then* $n + 1 = 4$.

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