

PARTIALIZATION OF CATEGORIES AND INVERSE BRAID-PERMUTATION MONOID

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ABSTRACT. We show how the categorial approach to inverse monoids can be described as a certain endofunctor (which we call the partialization functor) of some category. In this paper we show that this functor can be used to obtain several recently defined inverse monoids, and use it to define a new object, which we call the inverse braid-permutation monoid. A presentation for this monoid is obtained. Finally, we study some abstract properties of the partialization functor and its iterations. This leads to a categorification of a monoid of all order preserving maps, and series of orthodox generalizations of the symmetric inverse semigroup.

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1. INTRODUCTION

The study of inverse semigroups forms a classical part of the theory of semigroups. Several approaches to construction of inverse semigroups are known. The most abstract one seems to be the categorical approach, which was worked out in various variations in [Ho, Le, La]. In Section 2 we give a short overview of this approach in the interpretation, most suitable for our further purposes. Roughly speaking, starting with a small category with pullbacks for monomorphisms, there is a functorial way to enlarge the set of morphisms of this category by what is naturally to call “partial morphisms”. This defines a functor, which we call the *partialization functor*. Under some mild assumptions the endomorphism monoids of the new category turn out to be inverse monoids. For example, starting from the category of all finite sets in which the morphisms are all monomorphisms, the described above partialization procedure gives us a new category, where the endomorphism monoids are just symmetric inverse monoids. Several other examples are discussed in Section 3.

One can apply this approach to produce several inverse monoids which recently appeared in the literature. The main example which we have in mind is the inverse braid monoid, defined and studied in [EL]. The idea is the following: the braid group can be realized as the mapping class group (of homeomorphisms with compact support) of a punctured plane. In Section 4 we define the category, whose objects are punctured planes with different punctures and whose morphism sets are isotopy classes of homeomorphisms between such planes. It turns out that the procedure of categorical partialization is directly applicable. The result is a new category, in which the endomorphism monoids are exactly the inverse braid monoids from [EL].

The latter example has an immediate generalization. Instead of the punctured plane let us consider the space of n unknotted and unlinked simple closed oriented curves in \mathbb{R}^3 . The corresponding motion group is the so-called *braid-permutation group*, defined in [Da] and studied in [Wa, Ru, Mc, FRR]. The categorical partialization is again directly applicable and produces a category, in which it is natural to call endomorphism monoids the *inverse braid-permutation monoids*. These monoids are new. We describe all constructions necessary for its definition in Section 5, where we also obtain a presentation for this monoid.

Finally, in Section 6 we go back to the abstract study of the partialization functor. We show that this functor is in fact an endofunctor of a certain category. We explicitly describe its iterations and show how they are connected with each other by some canonical natural transformations. In this way we obtain a categorification of the monoid of all order preserving maps on a chain, which fix the endpoints. We also describe certain quasi-iterations of the partialization functor on a somewhat bigger category. In natural examples the iterations of the partialization functor produce orthodox monoids. We pay special attention to orthodox generalizations of the symmetric inverse monoid.

For a monoid, S we denote by $E(S)$ the set of all idempotents of S and by $G(S)$ the group of units of S . Green's relations on S are denoted by \mathcal{L} , \mathcal{R} , \mathcal{H} , \mathcal{D} , and \mathcal{J} . Recall that a monoid S is called *factorizable* if $S = E(S)G(S)$.

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2. PARTIALIZATIONS OF A CATEGORY

In this section we recall the partialization procedure for small categories, worked out in [Ho, Le, La]. However, our setup is slightly different from the one available in the literature. It is more adapted to our further needs.

2.1. The partialization functor \mathcal{P} . In what follows \mathcal{C} is a small category. For $i, j \in \mathcal{C}$ we denote by $\mathcal{C}(i, j)$ the set of morphisms from i to j . For $i \in \mathcal{C}$ we denote by 1_i the identity morphisms in $\mathcal{C}(i, i)$.

Assume that \mathcal{C} satisfies the following condition:

Condition 1. For any $i, j, k \in \mathcal{C}$, any $f \in \mathcal{C}(i, j)$, and any monomorphism $\alpha \in \mathcal{C}(k, j)$ there exists a pullback diagram

$$\begin{array}{ccc} i & \xrightarrow{f} & j \\ \uparrow \beta & & \uparrow \alpha \\ m & \xrightarrow{\hat{f}} & k \end{array}$$

We are now going to define a new category, $\mathcal{P}(\mathcal{C})$, as follows: $\mathcal{P}(\mathcal{C})$ has the same objects as \mathcal{C} . For $i, j \in \mathcal{P}(\mathcal{C})$ to define $\mathcal{P}(\mathcal{C})(i, j)$ we consider all possible diagrams of the form

$$(2.1) \quad D(i, j, k, \alpha, f) : \quad i \xleftarrow{\alpha} k \xrightarrow{f} j \quad \text{or} \quad \begin{array}{ccc} & & i \\ & & \uparrow \alpha \\ & & k \xrightarrow{f} j \end{array}$$

We will say that the diagrams $D(i, j, k, \alpha, f)$ and $D(i', j', k', \alpha', f')$ are equivalent provided that $i' = i$, $j' = j$, and there is an isomorphism, γ , which makes the following diagram commutative:

$$(2.2) \quad \begin{array}{ccccc} i & \xleftarrow{\alpha} & k & \xrightarrow{f} & j \\ \parallel & & \uparrow \gamma^{-1} & & \parallel \\ i & \xleftarrow{\alpha'} & k' & \xrightarrow{f'} & j \\ \parallel & & \downarrow \gamma & & \parallel \end{array}$$

Denote by $\mathcal{P}(\mathcal{C})(i, j)$ the set of all equivalence classes. We define the composition of $D(i, j, k, \alpha, f)$ with $D(j, m, n, \beta, g)$ as $D(i, m, l, \alpha\gamma, gh)$ via the following pullback diagram, whose existence is guaranteed by Condition 1:

$$(2.3) \quad \begin{array}{ccccc} & & i & & \\ & & \uparrow \alpha & & \\ & & k & \xrightarrow{f} & j \\ & & \uparrow \gamma & & \uparrow \beta \\ & & l & \xrightarrow{h} & n \xrightarrow{g} m \end{array}$$

It is straightforward to verify that the above composition is well-defined and associative and hence $\mathcal{P}(\mathcal{C})$ is a category.

For $\mathbf{i} \in \mathcal{C}$ set $\mathbf{i}(\mathbf{i}) = \mathbf{i} \in \mathcal{P}(\mathcal{C})$; and for $f \in \mathcal{C}(\mathbf{i}, \mathbf{j})$ set $\mathbf{i}(f) = D(\mathbf{i}, \mathbf{j}, \mathbf{i}, 1_{\mathbf{i}}, f)$. This obviously defines a faithful functor, $\mathbf{i} : \mathcal{C} \rightarrow \mathcal{P}(\mathcal{C})$, which we will call the *canonical inclusion*.

Let now \mathcal{C} and \mathcal{C}' be two categories satisfying Condition 1 and let $F : \mathcal{C} \rightarrow \mathcal{C}'$ be a functor, which preserves monomorphisms. Then F maps the diagram $D(\mathbf{i}, \mathbf{j}, \mathbf{k}, \alpha, f)$ to $D(F(\mathbf{i}), F(\mathbf{j}), F(\mathbf{k}), F(\alpha), F(f))$ and the functorial properties of F ensure that the equivalent diagrams end up in equivalent diagrams. Hence F induces a functor from $\mathcal{P}(\mathcal{C})$ to $\mathcal{P}(\mathcal{C}')$, which we will denote by $\mathcal{P}(F)$. In particular, \mathcal{P} becomes a functor from the category of all small categories, satisfying Condition 1, where morphisms are all functors, preserving monomorphisms, to the category of all small categories. We will call \mathcal{P} the *(first) partialization functor*. The functor \mathbf{i} , when applied to all possible categories satisfying Condition 1, gives a natural transformation from the canonical inclusion functor to \mathcal{P} .

2.2. \mathcal{P} and inverse semigroups. From now on we always assume that \mathcal{C} is a small category satisfying Condition 1. The connection of \mathcal{P} to inverse semigroups is given by the following statement:

Proposition 2. *Suppose that all morphisms in \mathcal{C} are monomorphisms. Then $\text{End}_{\mathcal{P}(\mathcal{C})}(\mathbf{i})$ is an inverse semigroup for every object $\mathbf{i} \in \text{Ob } \mathcal{C}$.*

Proof. We start the proof with the following easy observation, which one proves by a direct calculation:

Lemma 3. *Let $\mathbf{i}, \mathbf{j}, \mathbf{l} \in \mathcal{C}$. Let further $\alpha \in \mathcal{C}(\mathbf{l}, \mathbf{i})$ and $\beta \in \mathcal{C}(\mathbf{l}, \mathbf{j})$ be monomorphisms. Then the elements $x = D(\mathbf{i}, \mathbf{j}, \mathbf{l}, \alpha, \beta)$ and $y = D(\mathbf{j}, \mathbf{i}, \mathbf{l}, \beta, \alpha)$ satisfy $xyx = x$ and $yx = y$.*

It follows from Lemma 3 that $\text{End}_{\mathcal{P}(\mathcal{C})}(\mathbf{i})$ is a regular semigroup. Hence, to prove our proposition we have just to show that the idempotents of $\text{End}_{\mathcal{P}(\mathcal{C})}(\mathbf{i})$ commute. We proceed by describing the idempotents:

Lemma 4. *The element $x = D(\mathbf{i}, \mathbf{i}, \mathbf{l}, \alpha, \beta) \in \mathcal{P}(\mathcal{C})(\mathbf{i}, \mathbf{i})$ is an idempotent if and only if $\alpha = \beta$.*

Proof. If $\alpha = \beta$, then $x^2 = x$ is checked by a direct calculation. Consider the pullback diagram

$$(2.4) \quad \begin{array}{ccc} \mathbf{l} & \xrightarrow{\beta} & \mathbf{i} \\ \alpha' \uparrow & & \uparrow \alpha \\ \mathbf{l}' & \xrightarrow{\beta'} & \mathbf{l} \end{array}$$

Then $x^2 = D(\mathbf{i}, \mathbf{i}, \mathbf{l}', \alpha\alpha', \beta\beta')$. Using (2.2), the equivalence of x^2 and x implies the existence of an isomorphism $\gamma \in \mathcal{C}(\mathbf{l}', \mathbf{l})$ such that $\alpha\gamma = \alpha\alpha'$ and $\beta\gamma = \beta\beta'$. Since α and β are monomorphisms, the latter equalities imply $\gamma = \alpha' = \beta'$, in particular, both α' and β' are isomorphisms. Now (2.4) implies $\alpha = \beta$. \square

Let now $x = D(\mathbf{i}, \mathbf{i}, \mathbf{l}, \alpha, \alpha)$ and $y = D(\mathbf{i}, \mathbf{i}, \mathbf{l}', \beta, \beta)$ and

$$\begin{array}{ccc} \mathbf{l}' & \xrightarrow{\beta} & \mathbf{i} \\ \delta \uparrow & & \uparrow \alpha \\ \mathbf{t} & \xrightarrow{\gamma} & \mathbf{l} \end{array}$$

be the pullback. Then $\alpha\gamma = \beta\delta$ and a direct calculation imply $xy = yx$. This completes the proof. \square

2.3. The partialization functor \mathcal{Q} . Let \mathcal{C} be as in Subsection 2.1. For $i, j \in \mathcal{C}$ denote by $\mathcal{Q}(\mathcal{C})(i, j)$ the subset of $\mathcal{P}(\mathcal{C})(i, j)$ consisting of all $D(i, j, k, \alpha, f)$ for which there exists $\bar{f} \in \mathcal{C}(i, j)$ making the following diagram commutative:

$$(2.5) \quad \begin{array}{ccc} & i & \\ & \uparrow \alpha & \searrow \bar{f} \\ k & \xrightarrow{f} & j. \end{array}$$

It is easy to see that $\mathcal{Q}(\mathcal{C})$ is a subcategory of $\mathcal{P}(\mathcal{C})$ and that the natural inclusion $\mathcal{C} \hookrightarrow \mathcal{P}(\mathcal{C})$ factors through $\mathcal{Q}(\mathcal{C})$. From this in the same way as in Subsection 2.1 one gets that \mathcal{Q} defines a functor, which we will call the *(second) partialization functor*. The natural transformation i then factors as $i = i''i'$, where i' is the natural transformation from the natural inclusion to \mathcal{Q} , and i'' is the natural transformation (inclusion) from \mathcal{Q} to \mathcal{P} .

The “philosophical” difference between \mathcal{P} and \mathcal{Q} is how one understands the notion of a *partial map*: for \mathcal{P} a partial map is a map, defined on a subobject; whereas for \mathcal{Q} a partial map is a restriction of an ordinary map to a subobject. Both notions have their advantages and disadvantages, which could be seen for example after comparing Subsection 2.2 and Subsection 2.4.

The element in (2.5) will be denoted by $\bar{D}(i, j, k, \alpha, f, \bar{f})$. Note that $f = \bar{f}\alpha$ by definition.

2.4. \mathcal{Q} and inverse semigroups.

Proposition 5. *Suppose that \mathcal{C} is as above and $i \in \text{Ob } \mathcal{C}$ is such that $\text{End}_{\mathcal{C}}(i)$ is a group. Then the semigroup $\text{End}_{\mathcal{Q}(\mathcal{C})}(i)$ is an inverse monoid.*

Proof. We will show that $\text{End}_{\mathcal{Q}(\mathcal{C})}(i)$ is a regular monoid with commuting idempotents.

Lemma 6. $\text{End}_{\mathcal{Q}(\mathcal{C})}(i)$ is regular.

Proof. Let $x = \bar{D}(i, i, k, \alpha, f, g)$, where $f = g\alpha$, be an element of $\mathcal{Q}(\mathcal{C})(i, i)$. Since $\text{End}_{\mathcal{C}}(i)$ is a group, we can consider $y = \bar{D}(i, i, i, 1_i, g^{-1}, g^{-1})$. A direct calculation shows that $xyx = x$ and hence x is regular. This completes the proof. \square

Lemma 7.

$$E(\text{End}_{\mathcal{Q}(\mathcal{C})}(i)) = \{\bar{D}(i, i, k, \alpha, \alpha, 1_i)\}.$$

Proof. That each $\bar{D}(i, i, k, \alpha, \alpha, 1_i)$ is an idempotent is obtained by a direct calculation.

Now let $x = \bar{D}(i, i, k, \alpha, f, g)$, $g = f\alpha$ be an idempotent. Then $i''(x) = D(i, i, k, \alpha, f)$ is an idempotent as well. Since g is an isomorphism, f is a monomorphism and hence Lemma 4 implies that without loss of generality we can assume $\alpha = f$. The statement follows. \square

Now a direct calculation shows that for $x, y \in \{\bar{D}(i, i, k, \alpha, \alpha, 1_i)\}$ we have $xy = yx$. This completes the proof of the proposition. \square

Corollary 8. *Assume all conditions of Proposition 5.*

- (i) *The monoid $\text{End}_{\mathcal{Q}(\mathcal{C})}(i)$ is factorizable.*
- (ii) *$\text{End}_{\mathcal{Q}(\mathcal{C})}(i)$ is a maximal factorizable subset of $\text{End}_{\mathcal{P}(\mathcal{C})}(i)$.*
- (iii) *$\text{End}_{\mathcal{Q}(\mathcal{C})}(i) = \text{End}_{\mathcal{P}(\mathcal{C})}(i)$ if and only if $\text{End}_{\mathcal{P}(\mathcal{C})}(i)$ is factorizable.*

Proof. (i) follows from the definition of \mathcal{Q} . From the proofs of Proposition 2 and Proposition 5 we have

$$E(\text{End}_{\mathcal{Q}(\mathcal{C})}(\mathbf{i})) = E(\text{End}_{\mathcal{P}(\mathcal{C})}(\mathbf{i})) \quad \text{and} \quad G(\text{End}_{\mathcal{Q}(\mathcal{C})}(\mathbf{i})) = G(\text{End}_{\mathcal{P}(\mathcal{C})}(\mathbf{i})).$$

This and (i) imply (ii), and (iii) follows from (ii). \square

2.5. Dual constructions. All the constructions above admit obvious dualization (in other words, the dualization reduces the dual constructions to the two constructions described above). The semigroup-theoretical effect of this is that one obtains the opposite semigroup (category). This might look trivial, but in fact it is not. Natural categories might have rather non-symmetric structure. Thus the category of all finite sets, where morphisms are all injections of sets, is **not** dual to the category of all finite sets, where morphisms are all surjections of sets. The latter category satisfies the condition of pushouts for epimorphisms, and hence its opposite category satisfies Condition 1. Applied to these two very different categories, our construction produces different inverse monoids. This will be discussed later on in examples.

3. EXAMPLES FROM THE THEORY OF SEMIGROUPS

In this section we show that many classical examples of inverse semigroups can be obtained using \mathcal{P} or \mathcal{Q} .

3.1. Finite symmetric inverse semigroup. Let \mathcal{C}_1 denote the category, whose objects are finite sets and morphisms are injective maps. In particular, all morphisms in this category are monomorphisms. If $X, Y, Z \in \mathcal{C}_1$, $f : X \rightarrow Y$, $g : Z \rightarrow Y$, we can define $U = \{x \in X : f(x) \in g(Z)\}$ and for $u \in U$ define $h(u)$ as the unique element of Z such that $f(u) = g(h(u))$. The map h is obviously an injection. One shows that the commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \text{incl} \uparrow & & \uparrow g \\ U & \xrightarrow{h} & Z \end{array}$$

is a pullback. Hence \mathcal{C}_1 satisfies Condition 1. For $X \in \mathcal{C}_1$ we have that $\text{End}_{\mathcal{C}_1}(X)$ is the symmetric group on X . A direct computation shows that $\text{End}_{\mathcal{P}(\mathcal{C}_1)}(X) = \text{End}_{\mathcal{Q}(\mathcal{C}_1)}(X)$ is the full symmetric inverse semigroup $\mathcal{IS}(X)$ on X . Note that, although $\text{End}_{\mathcal{P}(\mathcal{C}_1)}(X) = \text{End}_{\mathcal{Q}(\mathcal{C}_1)}(X)$ for each $X \in \mathcal{C}_1$, the categories $\mathcal{P}(\mathcal{C}_1)$ and $\mathcal{Q}(\mathcal{C}_1)$ are different: if $X, Y \in \mathcal{C}_1$ and $0 < |X| < |Y|$, then $\mathcal{P}(\mathcal{C}_1)(Y, X) \neq \emptyset$ while $\mathcal{Q}(\mathcal{C}_1)(Y, X) = \emptyset$.

3.2. Dual symmetric inverse semigroup. Let \mathcal{C}_2 denote the category, whose objects are finite sets and morphisms are surjective maps. In particular, we have that all morphisms in this category are epimorphisms. One shows that the opposite category $\mathcal{C}_2^{\text{op}}$ satisfies Condition 1. For $X \in \mathcal{C}_1$ we have that $\text{End}_{\mathcal{C}_1}(X)$ is the symmetric group on X . A direct computation shows that $\text{End}_{\mathcal{P}(\mathcal{C}_2^{\text{op}})}(X)$ is the dual symmetric inverse monoid \mathcal{I}_X^* , and $\text{End}_{\mathcal{Q}(\mathcal{C}_2^{\text{op}})}(X)$ is the greatest factorizable submonoid \mathcal{F}_X^* of \mathcal{I}_X^* , see [FL].

3.3. The semigroup \mathcal{PT}_n of all partial transformations. Let \mathcal{C}_3 denote the category, whose objects are all finite sets and morphisms are all maps between these sets. One shows that \mathcal{C}_3 satisfies Condition 1. For $X \in \mathcal{C}_3$ we have that $\text{End}_{\mathcal{C}_3}(X) = \mathcal{T}(X)$ is the full transformation semigroup on X . A direct calculation shows that $\text{End}_{\mathcal{P}(\mathcal{C}_3)}(X)$ is the semigroup $\mathcal{PT}(X)$ of all partial transformations

on X . One can consider the dual situation (as in Subsection 3.2). It might be interesting to try to understand the resulting monoid.

3.4. Partial endomorphisms of a group. Consider the category \mathcal{C}_4 , whose objects are finite groups and morphisms are group homomorphisms. One shows that \mathcal{C}_4 satisfies Condition 1. If G is a finite group, the monoid $\text{End}_{\mathcal{C}_4}(G)$ is the monoid of all endomorphisms of G . The monoid $\text{End}_{\mathcal{P}(\mathcal{C}_4)}(G)$ is the monoid of all partial endomorphisms of G , see [NN]. Note that $\text{End}_{\mathcal{P}(\mathcal{C}_4)}(G) \neq \text{End}_{\mathcal{Q}(\mathcal{C}_4)}(G)$ in general.

3.5. Partial automorphisms of a group. Consider the category \mathcal{C}_5 , whose objects are finite groups and morphisms are group monomorphisms. One shows that \mathcal{C}_5 satisfies Condition 1. If G is a finite group, $\text{End}_{\mathcal{C}_5}(G)$ is the group of all automorphisms of G . The monoid $\text{End}_{\mathcal{P}(\mathcal{C}_5)}(G)$ is the inverse monoid of all partial automorphisms of G . The dual construction (using epimorphisms) gives the monoid of bicongruences of a group, defined in [Fi] (even in a more general setup of universal algebras).

3.6. Partial linear endomorphisms. Let \mathbb{k} be a field. Denote by \mathcal{C}_6 the category, whose objects are \mathbb{k}^n , $n = 0, 1, \dots$, and morphisms are all linear maps. One shows that \mathcal{C}_6 satisfies Condition 1. The monoid $\text{End}_{\mathcal{C}_6}(\mathbb{k}^n)$ is the monoid of all $n \times n$ matrices over \mathbb{k} . The monoid $\text{End}_{\mathcal{P}(\mathcal{C}_6)}(\mathbb{k}^n)$ is the monoid of all partial linear maps on \mathbb{k}^n .

3.7. Partial linear automorphisms. Denote by \mathcal{C}_7 the category, whose objects are \mathbb{k}^n , $n = 0, 1, \dots$, and morphisms are all injective linear maps. One shows that \mathcal{C}_7 satisfies Condition 1. The monoid $\text{End}_{\mathcal{C}_7}(\mathbb{k}^n)$ is the group of all invertible $n \times n$ matrices over \mathbb{k} . The inverse monoid $\text{End}_{\mathcal{P}(\mathcal{C}_7)}(\mathbb{k}^n)$ is the monoid of all partial linear automorphisms of \mathbb{k}^n , studied in e.g. [Ku].

3.8. Partially defined co-finite automorphisms of integers. Consider \mathbb{Z} as a metric space with respect to the usual metric $\rho(m, n) = |m - n|$. Let \mathcal{C}_8 be the category, whose objects are all co-finite subsets of \mathbb{Z} and morphisms are all isometries. One shows that \mathcal{C}_8 satisfies Condition 1. The monoid $\text{End}_{\mathcal{C}_8}(\mathbb{Z})$ is the group of all isometries of \mathbb{Z} , which is in fact isomorphic to the infinite dihedral group. The inverse monoid $\text{End}_{\mathcal{P}(\mathcal{C}_8)}(\mathbb{Z})$ is the inverse monoid of all partially defined co-finite automorphisms of integers, studied in [Be].

3.9. Leech's approach to inverse monoids. Let \mathcal{C}_9 be an *abstract division category* with *initial object* I in the sense of [Le, 1.1]. Then $\mathcal{C}_9^{\text{op}}$ satisfies Condition 1. The monoid $\text{End}_{\mathcal{P}(\mathcal{C}_9^{\text{op}})}(I)$ is isomorphic to the inverse monoid associated with \mathcal{C}_9 as defined in [Le, 2.1]. In particular, if \mathcal{C}_9 has one object and all morphisms in \mathcal{C}_9 are monomorphisms, then Condition 1 describes exactly the situation, dual to the one discussed in [Le, 1.2].

3.10. Bisimple inverse monoid. Let \mathcal{C}_{10} denote the category, whose objects are cofinite subsets of \mathbb{N} and morphisms are all possible injections with cofinite image (in particular, all objects of \mathcal{C}_{10} are isomorphic). In the same way as for \mathcal{C}_1 one shows that \mathcal{C}_{10} satisfies Condition 1. The monoid $\text{End}_{\mathcal{P}(\mathcal{C}_{10}^{\text{op}})}(\mathbb{N})$ is a bisimple inverse monoid.

4. THE INVERSE BRAID MONOID

In this section we give one more example. The semigroup we will talk about is the *inverse braid monoid*, recently defined in [EL]. But our approach to this monoid will be quite different. We will obtain this monoid as one more application of the functor \mathcal{P} .

Consider the category \mathcal{B} defined as follows: The objects of \mathcal{B} are indexed by finite subsets of \mathbb{N} . If $X \subset \mathbb{N}$, then the object, associated with X , is the plane $P_X = \mathbb{R}^2$ with marked points $(i, 0)$, $i \in X$. For $X, Y \subset \mathbb{N}$, $|X|, |Y| \leq \infty$, the set $\mathcal{B}(P_X, P_Y)$ is the set of all isotopy classes of homeomorphisms with compact support from P_X to P_Y , which map marked points to marked points. The composition is induced by the usual composition of maps. In particular, if $|X| > |Y|$ then $\mathcal{B}(P_X, P_Y) = \emptyset$; if $|X| = |Y| = n$ then $\mathcal{B}(P_X, P_Y)$ is identified (elementwise) with Artin's braid group \mathcal{B}_n , see [Bi, Theorem 1.10]; if $|X| < |Y|$ we have $\binom{|Y|}{|X|}$ ways to choose the set A of values for the marked points from P_X and after fixing it the part of $\mathcal{B}(P_X, P_Y)$ corresponding to A is again identified (elementwise) with $\mathcal{B}_{|X|}$ via [Bi, Theorem 1.10]. We also have that $\mathcal{B}(P_X, P_X) = \mathcal{B}_{|X|}$ as a group.

First we note that obviously all morphisms in \mathcal{B} are monomorphisms. Further, \mathcal{B} satisfies Condition 1. Indeed, let $f \in \mathcal{B}(P_X, P_Y)$ and $g \in \mathcal{B}(P_Z, P_Y)$. We have $|X| \leq |Y|$ and $|Z| \leq |Y|$. Let A and B denote the set of marked points in P_Y , which are images of mark points from P_X and P_Z under f and g respectively. Let $C = A \cap B$, $l = |C|$, and $V \subset \mathbb{N}$ be such that $|V| = l$. Let $h \in \mathcal{B}(P_V, P_X)$ be any map, which sends the marked points to $f^{-1}(C)$. Consider the set \mathcal{S} of all morphisms from $\mathcal{B}(P_V, P_Z)$, which send marked points to $g^{-1}(C)$. Then $g\mathcal{S}$ is identified with \mathcal{B}_l via [Bi, Theorem 1.10] and hence there exists a unique $h' \in \mathcal{S}$ such that $gh' = fh$. One now easily shows that the diagram

$$\begin{array}{ccc} P_X & \xrightarrow{f} & P_Y \\ \uparrow h & & \uparrow g \\ P_V & \xrightarrow{h'} & P_Z \end{array}$$

is a pullback. This implies Condition 1. In particular, we have the partialization $\mathcal{P}(\mathcal{B})$ and from Proposition 2 we get that the monoid $\text{End}_{\mathcal{P}(\mathcal{B})}(P_X)$ is an inverse monoid.

Let $X = \{1, 2, \dots, n\}$. We now claim that $\text{End}_{\mathcal{P}(\mathcal{B})}(P_X)$ is isomorphic to the inverse braid monoid \mathcal{IB}_n as defined in [EL]. The monoid \mathcal{IB}_n is defined as the set of geometrical braids with n strands in which some strands can be missing with the obvious multiplication induced by the multiplication of geometrical braids. Let us construct a bijection from \mathcal{IB}_n to $\text{End}_{\mathcal{P}(\mathcal{B})}(P_X)$ as follows: Denote the base points for geometric braids by $1, 2, \dots, n$. Take some partial geometrical braid $\alpha \in \mathcal{IB}_n$. Assume that α consists of m strands. Then α is given by two subsets $A, B \subset \{1, 2, \dots, n\}$ and a usual braid $\bar{\alpha}$ on m strands, namely, $\bar{\alpha}$ connects initial points from A with the terminal points from in B .

The class of the identity transformation of \mathbb{R}^2 is a morphism in $\mathcal{B}(P_A, P_X)$, which we denote by $i(A, X)$. Let $g \in \mathcal{B}(P_A, P_X)$ be the isotopy class of maps, which maps $\{(j, 0) : j \in A\}$, to $\{(k, 0) : k \in B\}$ and corresponds to $\bar{\alpha}$ under the identification, given by [Bi, Theorem 1.10]. Then it follows immediately that the map

$$\begin{aligned} \mathcal{IB}_n &\rightarrow \text{End}_{\mathcal{P}(\mathcal{B})}(P_X) \\ \alpha &\mapsto P_X \xleftarrow{i(A, X)} P_A \xrightarrow{g} P_X \end{aligned}$$

is in fact an isomorphism of monoids.

5. THE INVERSE BRAID-PERMUTATION MONOID

In this section we generalize the example from the previous section and construct a new inverse monoid with topological origin. Again our construction is an immediate application of the functor \mathcal{P} .

5.1. Definition. Consider the category $\overline{\mathcal{B}}$ defined as follows: The objects of $\overline{\mathcal{B}}$ are indexed by finite subsets of \mathbb{N} . If $X \subset \mathbb{N}$, then the object, associated with X , is the space $Q_X = \mathbb{R}^3$ with marked circles $\{(\cos(a), \sin(a), i) : a \in [0, 2\pi)\}$, $i \in X$, with the orientation induced by the natural order on $[0, 2\pi)$. For $X, Y \subset \mathbb{N}$, $|X|, |Y| \leq \infty$, the set $\overline{\mathcal{B}}(Q_X, Q_Y)$ is the set of all isotopy classes of diffeomorphisms with compact support from Q_X to Q_Y , which map marked circles to marked circles preserving the orientation. The composition is induced by the usual composition of maps. In particular, if $|X| > |Y|$ then $\overline{\mathcal{B}}(Q_X, Q_Y) = \emptyset$; if $|X| = |Y| = n$ then $\overline{\mathcal{B}}(Q_X, Q_Y)$ is identified (elementwise) with the braid-permutation group BP_n , see [Da, Wa] (see also [FRR, Ru]); if $|X| < |Y|$ we have $\binom{|Y|}{|X|}$ ways to choose the set A of marked circles in Q_Y for the values of the marked circles from Q_X and after fixing it the part of $\overline{\mathcal{B}}(Q_X, Q_Y)$ corresponding to A is again identified (elementwise) with BP_n . It follows that $\overline{\mathcal{B}}(Q_X, Q_X) = \text{BP}_{|X|}$ as a group.

Again all morphisms in $\overline{\mathcal{B}}$ are monomorphisms. In the same way as in the previous section one shows that $\overline{\mathcal{B}}$ satisfies Condition 1. Hence we can apply \mathcal{P} and by Proposition 2 we get that the monoid $\text{End}_{\mathcal{P}(\overline{\mathcal{B}})}(Q_X)$ is an inverse monoid. We call this monoid the *inverse braid-permutation monoid* and denote it by IBP_n , where $n = |X|$.

5.2. Idempotents and factorizability of IBP_n . Let $Y \subset X$. Then the class of the identity transformation of \mathbb{R}^3 is a monomorphism in $\overline{\mathcal{B}}(Q_Y, Q_X)$, which we denote by f_Y . Denote by ε_Y the element $Q_X \xleftarrow{f_Y} Q_Y \xrightarrow{f_Y} Q_X$ in IBP_n .

Lemma 9. $E(\text{IBP}_n) = \{\varepsilon_Y : Y \subset X\}$. In particular, $E(\text{IBP}_n)$ is canonically isomorphic to the Boolean $(2^X, \cap)$ of X .

Proof. Consider the element $x \in \text{IBP}_n$ given by $Q_X \xleftarrow{f} Q_Y \xrightarrow{f} Q_X$, where $f \in \overline{\mathcal{B}}(Q_Y, Q_X)$. Let $\mathbf{f} \in f$ and $Z = \mathbf{f}(Y)$. Then the class of \mathbf{f}^{-1} is an element of $\overline{\mathcal{B}}(Q_Z, Q_Y)$, call it g . Note that g is an isomorphism since $|Y| = |Z|$. It follows that the following diagram commutes:

$$\begin{array}{ccccc} Q_X & \xleftarrow{f} & Q_Y & \xrightarrow{f} & Q_X \\ \parallel & & \uparrow g & \downarrow g^{-1} & \parallel \\ Q_X & \xleftarrow{fz} & Q_Z & \xrightarrow{fz} & Q_X \end{array}$$

Hence Lemma 4 implies that $E(\text{IBP}_n) = \{\varepsilon_Y : Y \subset X\}$. A direct calculation shows that $\varepsilon_Y \mapsto Y$ is an epimorphism from $E(\text{IBP}_n)$ to $(2^X, \cap)$. Hence it is an isomorphism since $|E(\text{IBP}_n)| \leq 2^n$ by above. \square

By construction, we have

$$G(\text{IBP}_n) = \text{i}(\text{End}_{\overline{\mathcal{B}}}(Q_X)) \cong \text{BP}_n.$$

Lemma 10. IBP_n is a factorizable monoid.

Proof. Let $x \in \text{IBP}_n$. By the same arguments as in the proof of Lemma 9 we can assume that x is given by $Q_X \xleftarrow{f_Y} Q_Y \xrightarrow{f} Q_X$ for some $f \in \overline{\mathcal{B}}(Q_Y, Q_X)$. Choose a representative $\mathbf{f} \in f$, which preserves the set of marked circles, given by

X . Then we can also consider the class $g \in \overline{\mathcal{B}}(Q_X, Q_X)$ containing \mathbf{f} . The class g defines an element of $\text{BP}_n = G(\mathcal{IBP}_n)$, which we call y . A direct calculation shows that $x = y\varepsilon_Y$. This completes the proof. \square

5.3. Presentation of \mathcal{IBP}_n . A presentation for BP_n was obtained in [Mc, FRR]. A presentation for \mathcal{IBP}_n was obtained in [EL]. In this subsection we obtain a presentation for \mathcal{IBP}_n . In some sense it is a unification of the results of from [FRR] and [EL]. Our arguments are based on the approach to presentations of factorizable inverse monoids, worked out in [EEF]. To formulate the result we will need to recall the presentation of BP_n , obtained in [FRR].

To simplify notation we assume $X = \{1, 2, \dots, n\}$. Let $i \in \{1, 2, \dots, n-1\}$. Denote by τ_i the element of BP_n given by a homeomorphism of \mathbb{R}^3 , whose support contains only the i -th and the $(i+1)$ -st circles, and which interchanges these two circles without moving them through each other. Denote by σ_i the element of BP_n given by a homeomorphism of \mathbb{R}^3 , whose support contains only the i -th and the $(i+1)$ -st circles, and which interchanges these two circles by moving the i -th circle through the $(i+1)$ -st (see [Ru, Section 2]). It turns out that $\{\sigma_i, \tau_i : i = 1, \dots, n-1\}$ is a generating set for BP_n . We even have:

Theorem 11. ([FRR]) *The group BP_n is generated by $\{\sigma_i, \tau_i : i = 1, \dots, n-1\}$ with the following defining relations:*

$$(5.1) \quad (\text{braid group relations}) \quad \begin{cases} \sigma_i \sigma_j & = \sigma_j \sigma_i, \quad |i-j| > 1; \\ \sigma_i \sigma_{i+1} \sigma_i & = \sigma_{i+1} \sigma_i \sigma_{i+1}; \end{cases}$$

$$(5.2) \quad (\text{permutation group relations}) \quad \begin{cases} \tau_i^2 & = 1; \\ \tau_i \tau_j & = \tau_j \tau_i, \quad |i-j| > 1; \\ \tau_i \tau_{i+1} \tau_i & = \tau_{i+1} \tau_i \tau_{i+1}; \end{cases}$$

$$(5.3) \quad (\text{mixed relations}) \quad \begin{cases} \sigma_i \tau_j & = \tau_j \sigma_i, \quad |i-j| > 1; \\ \tau_i \tau_{i+1} \sigma_i & = \sigma_{i+1} \tau_i \tau_{i+1}; \\ \sigma_i \sigma_{i+1} \tau_i & = \tau_{i+1} \sigma_i \sigma_{i+1}. \end{cases}$$

For $i = 1, \dots, n$ denote by ε_i the element ε_Y , where $Y = X \setminus \{i\}$. We have the following:

Theorem 12. *The monoid \mathcal{IBP}_n is generated by the elements $\{\sigma_i, \sigma_i^{-1}, \tau_i : i = 1, \dots, n-1\}$ and $\{\varepsilon_i : i = 1, \dots, n\}$ with the defining relations (5.1)–(5.3) and the following additional relations:*

$$(5.4) \quad (\text{inverse relation}) \quad \{ \sigma_i \sigma_i^{-1} = \sigma_i^{-1} \sigma_i = 1;$$

$$(5.5) \quad (\text{semilattice relations}) \quad \begin{cases} \varepsilon_i^2 & = \varepsilon_i; \\ \varepsilon_i \varepsilon_j & = \varepsilon_j \varepsilon_i; \end{cases}$$

$$(5.6) \quad (\text{action relations}) \quad \begin{cases} \sigma_i \varepsilon_j & = \varepsilon_j \sigma_i, \quad i \neq j, j+1; \\ \tau_i \varepsilon_j & = \varepsilon_j \tau_i, \quad i \neq j, j+1; \\ \sigma_i \varepsilon_i & = \varepsilon_{i+1} \sigma_i; \\ \sigma_i \varepsilon_{i+1} & = \varepsilon_i \sigma_i; \\ \tau_i \varepsilon_i & = \varepsilon_{i+1} \tau_i; \end{cases}$$

$$(5.7) \quad (\text{kernel relations}) \quad \begin{cases} \varepsilon_i \varepsilon_{i+1} \sigma_i & = \varepsilon_i \varepsilon_{i+1} \tau_i = \varepsilon_i \varepsilon_{i+1}; \\ \varepsilon_i \sigma_i^2 & = \varepsilon_i \sigma_i \tau_i = \varepsilon_i \end{cases}$$

Proof. The generators $\{\sigma_i, \sigma_i^{-1}, \tau_i : i = 1, \dots, n-1\}$ and relations (5.1)–(5.4) give a presentation for BP_n by Theorem 11. The relations (5.5) give a presentation for the semilattice $(\mathcal{B}(X), \cap)$. The relations (5.6), which are verified by a direct calculation, give an action of BP_n on $(\mathcal{B}(X), \cap)$.

According to [EEF, Theorem 6], the only relations, which are left are the ones of the form $eg = e$, where $e \in E(\mathcal{IBP}_n)$ and $g \in \text{BP}_n$. So, to complete the proof we have only to show that all these relations can be derived from the ones described above together with (5.7) (the latter ones are again easily verified by a direct calculation).

For $Y \subset X$ set $G_Y = \{g \in \text{BP}_n : e_Y g = e_Y\}$. By definition, the group G_Y consists of classes of diffeomorphisms with compact support from Q_X to Q_X , whose support does not intersect any of the circles, corresponding to Y . One easily shows that G_Y is generated by:

- (a) The canonical image of $\overline{\mathcal{B}}(Q_{X \setminus Y}, Q_{X \setminus Y})$ in BP_n , given by the natural inclusion.
- (b) Extra moves, which can be described as follows: Take two circles, the first one for $i \notin Y$ and the second one for $j \in Y$. Move the first circle through the second one and return it back without involving any other circles.

The group $\overline{\mathcal{B}}(Q_{X \setminus Y}, Q_{X \setminus Y})$ is isomorphic to BP_m , where $m = |X| - |Y|$ and hence is generated by the corresponding σ_i 's and τ_i 's. Hence in the equality $eg = e$ we can assume g to be one of these generators. Using conjugation by elements with BP_n (i.e. relations (5.6)) we thus reduce $eg = e$ to the first line of the kernel relations (5.7).

The extra moves described in (b) can be of two different kinds: after moving the first circle through the second one we can return it back with or without moving the second circle through the first one. This means that these extra moves are either of the form $x\sigma_i^2x^{-1}$ or of the form $x\sigma_i\tau_i x^{-1}$ (or $x\tau_i\sigma_i x^{-1}$), where $x \in \text{BP}_n$. Note that $\varepsilon_i\tau_i\sigma_i = \varepsilon_i$ follows from $\varepsilon_i\sigma_i\tau_i = \varepsilon_i$, $\tau_i^2 = 1$ and $\varepsilon_i\sigma_i^2 = \varepsilon_i$. Hence, up to conjugation by elements with BP_n (i.e. relations (5.6)) the condition $eg = e$, where g is our extra move, reduces to the second line of the kernel relations (5.7).

Now the statement of Theorem 12 follows immediately from [EEF, Theorem 6]. \square

Remark 13. The system of generators of \mathcal{IBP}_n , presented in Theorem 12 is reducible: relations (5.6) show that \mathcal{IBP}_n is already generated by $\{\sigma_i, \sigma_i^{-1}, \tau_i : i = 1, \dots, n-1\}$ and the element $\varepsilon = \varepsilon_1$. From Theorem 12 one easily derives that with respect to this irreducible system of generators the defining relations are (5.1)–(5.4) together with the following additional relations:

$$\begin{cases} \varepsilon^2 & = \varepsilon & = \varepsilon\sigma_1^2 & = \sigma_1^2\varepsilon & = \varepsilon\sigma_1\tau_1; \\ \varepsilon\sigma_i & = \sigma_i\varepsilon, & & i > 2; \\ \varepsilon\tau_i & = \tau_i\varepsilon, & & i > 2; \\ \varepsilon\sigma_1\varepsilon & = \varepsilon\sigma_1\varepsilon\sigma_1 & = \sigma_1\varepsilon\sigma_1\varepsilon. \end{cases}$$

Remark 14. The same approach as we used in Theorem 12 can be used to derive a presentation for \mathcal{IB}_n , substantially shortening the arguments from [EL].

6. ITERATIONS OF THE PARTIALIZATION FUNCTOR

6.1. \mathcal{P} and monomorphisms. To be able to iterate \mathcal{P} (or \mathcal{Q}) one has to ensure that $\mathcal{P}(\mathcal{C})$ (or $\mathcal{Q}(\mathcal{C})$ respectively) satisfies Condition 1. This is wrong in the general case, we will give an example in Subsection 6.4. The first important step to understand Condition 1 for $\mathcal{P}(\mathcal{C})$ ($\mathcal{Q}(\mathcal{C})$) is the following result, which describes monomorphisms in partialized categories:

Proposition 15. (a) Let $x = D(\mathbf{i}, \mathbf{j}, \mathbf{k}, \alpha, f)$. Then x is a monomorphism if and only if $x = \mathbf{i}(\gamma)$ for some monomorphism $\gamma \in \mathcal{C}(\mathbf{i}, \mathbf{j})$.
(b) Let $y = \overline{D}(\mathbf{i}, \mathbf{j}, \mathbf{k}, \alpha, f, g)$. Then y is a monomorphism if and only if $x = \mathbf{i}'(\gamma)$ for some monomorphism $\gamma \in \mathcal{C}(\mathbf{i}, \mathbf{j})$.

Proof. The statement (b) is a special case of the statement (a), so we prove (a). Suppose x is a monomorphism. Let $a = D(\mathbf{i}, \mathbf{i}, \mathbf{k}, \alpha, \alpha)$ and $b = D(\mathbf{i}, \mathbf{i}, \mathbf{i}, \mathbf{1}_i, \mathbf{1}_i)$. By a direct calculation one obtains $xa = xb = x$, implying that $a = b$. In particular, α is an isomorphism. Without loss of generality we hence can assume $x = \mathbf{i}(f)$. Let $g_1, g_2 \in \mathcal{C}(\mathbf{1}, \mathbf{i})$ be such that $g_1 \neq g_2$. Since \mathbf{i} is injective, we have $\mathbf{i}(g_1) \neq \mathbf{i}(g_2)$. Since x is a monomorphism in $\mathcal{P}(\mathcal{C})$ we get $\mathbf{i}(fg_1) = x\mathbf{i}(g_1) \neq x\mathbf{i}(g_2) = \mathbf{i}(fg_2)$. Hence $fg_1 \neq fg_2$. This implies that f is a monomorphism in \mathcal{C} .

Now let $f \in \mathcal{C}(\mathbf{i}, \mathbf{j})$ be a monomorphism and $x = \mathbf{i}(f)$. Let $c = D(\mathbf{k}, \mathbf{i}, \mathbf{1}, \alpha, g)$ and $d = D(\mathbf{k}, \mathbf{i}, \mathbf{1}', \beta, h)$ be such that $xc = xd$. Hence there exists an isomorphism $\gamma \in \mathcal{C}(\mathbf{1}, \mathbf{1}')$ such that the solid part of the following diagram commutes:

$$\begin{array}{ccccc} \mathbf{k} & \xleftarrow{\alpha} & \mathbf{1} & \xrightarrow{g} & \mathbf{i} & \xrightarrow{f} & \mathbf{j} \\ \parallel & & \uparrow \gamma^{-1} & & \vdots & & \parallel \\ \mathbf{k} & \xleftarrow{\beta} & \mathbf{1}' & \xrightarrow{h} & \mathbf{i} & \xrightarrow{f} & \mathbf{i} \end{array}$$

Since f is a monomorphism, the middle square of the diagram commutes as well, implying that c and d are equivalent. This means that x is a monomorphism. \square

6.2. Quasi-iterations of \mathcal{P} . Let \mathcal{C} be a category satisfying Condition 1. Then the category $\mathcal{P}(\mathcal{C})$ is well-defined, however, it does not have to satisfy Condition 1 (this problem will be addressed later on in this section). Anyway, we can just formally consider the morphisms in the ‘‘second partialization’’ $\mathcal{P}(\mathcal{P}(\mathcal{C}))$ as defined in (2.1). These will be elements $D(\mathbf{i}, \mathbf{j}, \mathbf{k}, \alpha, f)$, where f is a morphism from $\mathcal{P}(\mathcal{C})$, hence, in turn, have the form $D(\mathbf{k}, \mathbf{j}, \mathbf{1}, \beta, g)$ for some morphism g from \mathcal{C} . Because of Proposition 15, the element $D(\mathbf{i}, \mathbf{j}, \mathbf{k}, \alpha, f)$ can be viewed as the diagram

$$\mathbf{i} \xleftarrow{\alpha} \mathbf{k} \xleftarrow{\beta} \mathbf{1} \xrightarrow{g} \mathbf{j},$$

which consists only of elements from the original category \mathcal{C} . This suggests the following definition: Let $n \in \mathbb{N}$. We define the category $\mathcal{P}^{(k)}(\mathcal{C})$ as follows: The objects of $\mathcal{P}^{(k)}(\mathcal{C})$ are the same as the same objects as \mathcal{C} . For $\mathbf{i}, \mathbf{j} \in \mathcal{C}$ the set $\mathcal{P}^{(k)}(\mathcal{C})(\mathbf{i}, \mathbf{j})$ is the set of equivalence classes of diagrams

$$(6.1) \quad D(n, \mathbf{i}, \mathbf{j}, \mathbf{k}_i, \alpha_i, f) : \quad \mathbf{i} \xleftarrow{\alpha_n} \mathbf{k}_n \xleftarrow{\alpha_{n-1}} \dots \xleftarrow{\alpha_1} \mathbf{k}_1 \xrightarrow{f} \mathbf{j},$$

where $D(n, \mathbf{i}, \mathbf{j}, \mathbf{k}_i, \alpha_i, f)$ and $D(n, \mathbf{i}', \mathbf{j}', \mathbf{k}'_i, \alpha'_i, f')$ are said to be equivalent if there exist isomorphisms γ_i making the following diagram commutative:

$$(6.2) \quad \begin{array}{ccccccc} \mathbf{i} & \xleftarrow{\alpha_n} & \mathbf{k}_n & \xleftarrow{\alpha_{n-1}} & \dots & \xleftarrow{\alpha_1} & \mathbf{k}_1 & \xrightarrow{f} & \mathbf{j} \\ \parallel & & \uparrow \gamma_n^{-1} & & & & \uparrow \gamma_1^{-1} & & \parallel \\ \mathbf{i}' & \xleftarrow{\alpha'_n} & \mathbf{k}'_n & \xleftarrow{\alpha'_{n-1}} & \dots & \xleftarrow{\alpha'_1} & \mathbf{k}'_1 & \xrightarrow{f'} & \mathbf{j}' \end{array}$$

Define the composition of $D(n, \mathbf{i}, \mathbf{j}, \mathbf{l}_i, \alpha_i, f)$ and $D(n, \mathbf{j}, \mathbf{k}, \mathbf{m}_i, \beta_i, g)$ as the element $D(n, \mathbf{i}, \mathbf{k}, \mathbf{n}_i, \gamma_i, gf_1)$ where $\gamma_n = \alpha_n \dots \alpha_1 \gamma'_n$ from the following diagram, where all

squares all pullbacks:

$$(6.3) \quad \begin{array}{ccccccc} & & & & & & \mathbf{k} \\ & & & & & & \uparrow g \\ & & & & & & \mathbf{m}_1 \\ & & & & \xleftarrow{\beta_1} & & \uparrow f_1 \\ & & & \mathbf{m}_n & \xleftarrow{\beta_{n-1}} & \dots & \uparrow f_n \\ & & \xleftarrow{\beta_n} & & & & \uparrow f \\ \mathbf{j} & & & & & & \mathbf{n}_n \\ \uparrow f & & & & & & \uparrow \gamma'_n \\ \mathbf{i} & \xleftarrow{\alpha_n} & \mathbf{l}_n & \xleftarrow{\alpha_{n-1}} & \dots & \xleftarrow{\alpha_1} & \mathbf{l}_1 \\ & & & & & & \uparrow \gamma_{n-1} \\ & & & & & & \mathbf{n}_1 \\ & & & & & & \uparrow \gamma_1 \end{array}$$

It is straightforward to verify that the above composition is well-defined and associative. In particular, $\mathcal{P}^{(k)}(\mathcal{C})$ is a category. We have the canonical inclusion $i^{(k)} : \mathcal{C} \rightarrow \mathcal{P}^{(k)}(\mathcal{C})$, which is defined by sending $f \in \mathcal{C}(i, j)$ to the element $D(n, i, j, i, \mathbf{l}_i, f)$.

If \mathcal{C} and \mathcal{C}' are two categories, satisfying Condition 1 and $F : \mathcal{C} \rightarrow \mathcal{C}'$ is a functor, which preserves monomorphisms, then F defines a functor from $\mathcal{P}^{(k)}(\mathcal{C})$ to $\mathcal{P}^{(k)}(\mathcal{C}')$ by mapping $D(n, i, j, \mathbf{l}_i, \alpha_i, f)$ to $D(n, F(i), F(j), F(\mathbf{l}_i), F(\alpha_i), F(f))$. In particular, $\mathcal{P}^{(k)}$ becomes a functor from the category of all small categories, satisfying Condition 1, where morphisms are all functors, preserving monomorphisms, to the category of all small categories. We will call $\mathcal{P}^{(k)}$ the k -th quasi-iteration of $\mathcal{P}^{(k)}$. The functor $i^{(k)}$ defined above is a natural transformation from the canonical inclusion functor to $\mathcal{P}^{(k)}$. It is convenient to let $\mathcal{P}^{(0)}$ denote the identity functor (or the natural inclusion into the category of all small categories).

For $k \geq 0$ we define a natural transformation $j_k : \mathcal{P}^{(k)} \rightarrow \mathcal{P}^{(k+1)}$ via

$$\begin{array}{ccc} \mathbf{j} & \mapsto & \mathbf{j} \\ \uparrow f & & \uparrow f \\ \mathbf{i} \xleftarrow{\alpha_n} \mathbf{k}_n \xleftarrow{\alpha_{n-1}} \dots \xleftarrow{\alpha_1} \mathbf{k}_1 & & \mathbf{i} \xleftarrow{\alpha_n} \mathbf{k}_n \xleftarrow{\alpha_{n-1}} \dots \xleftarrow{\alpha_1} \mathbf{k}_1 \equiv \mathbf{k}_1 \end{array}$$

It is straightforward to verify that j_k is indeed a natural transformation, moreover, it is injective. We obviously have $j_0 = i$.

For $k > 1$ we define a natural transformation $p_k : \mathcal{P}^{(k)} \rightarrow \mathcal{P}^{(k-1)}$ via

$$\begin{array}{ccc} \mathbf{j} & \mapsto & \mathbf{j} \\ \uparrow f & & \uparrow f \\ \mathbf{i} \xleftarrow{\alpha_n} \mathbf{k}_n \xleftarrow{\alpha_{n-1}} \dots \xleftarrow{\alpha_1} \mathbf{k}_1 & & \mathbf{i} \xleftarrow{\alpha_n} \mathbf{k}_n \xleftarrow{\alpha_{n-1}} \dots \xleftarrow{\alpha_3} \mathbf{k}_3 \xleftarrow{\alpha_2 \alpha_1} \mathbf{k}_1 \end{array}$$

It is straightforward to verify that p_k is indeed a natural transformation, moreover, it is surjective.

We have the following picture of functors and natural transformations:

$$(6.4) \quad \begin{array}{ccccccc} \mathcal{P}^{(0)} & \xrightarrow{j_0} & \mathcal{P}^{(1)} & \xleftarrow{p_2} & \mathcal{P}^{(2)} & \xleftarrow{p_3} & \mathcal{P}^{(3)} & \xleftarrow{p_4} & \dots \\ & & \uparrow j_1 & & \uparrow j_2 & & \uparrow j_3 & & \end{array}$$

Furthermore, we have the following easy fact, which follows directly from the definitions:

Lemma 16. *For each $k \geq 1$ the composition $p_{k+1}j_k$ is the identity.*

Remark 17. Although (6.4) and Lemma 16 is enough for our purposes, which will be explained in the next subsections, the real structure of $\{\mathcal{P}^{(k)}\}$ is richer. In fact, for each $k \geq 1$ and each $s, 1 \leq s \leq k$, we can define the natural transformation $j_k^{(s)} : \mathcal{P}^{(k)} \rightarrow \mathcal{P}^{(k+1)}$ as follows: In the element $D(n, i, j, \mathbf{k}_i, \alpha_i, f)$ we substitute the fragment \mathbf{k}_s with the fragment $\mathbf{k}_s \equiv \mathbf{k}_s$. Further, for each $k \geq 2$ and each

s , $2 \leq s \leq k$, we can define the natural transformation $\mathbf{p}_k^{(s)} : \mathcal{P}^{(k)} \rightarrow \mathcal{P}^{(k-1)}$ as follows: In the element $D(n, \mathbf{i}, \mathbf{j}, \mathbf{k}_i, \alpha_i, f)$ we skip \mathbf{k}_s and let the map from \mathbf{k}_{s-1} to \mathbf{k}_{s+1} be $\alpha_s \alpha_{s-1}$ (here by \mathbf{k}_{k+1} we mean \mathbf{i}). In particular, in the above notation we have $\mathbf{j}_k = \mathbf{j}_k^{(1)}$ and $\mathbf{p}_k = \mathbf{p}_k^{(2)}$. For all k and all appropriate s we have that the compositions $\mathbf{p}_{k+1}^{(s)} \cdot \mathbf{j}_k^{(s)}$ and $\mathbf{p}_{k+1}^{(s+1)} \cdot \mathbf{j}_k^{(s)}$ are identities. We will describe in more details the analogous structure for \mathcal{P}^k later on in Subsection 6.7. In fact the analogous structure for \mathcal{P}^k is even richer.

6.3. $\mathcal{P}^{(k)}$ and regular semigroups.

Theorem 18. *Let \mathcal{C} be a category in which all morphisms are monomorphisms and which satisfies Condition 1.*

- (i) *For each $k \geq 0$ and for each $\mathbf{i} \in \mathcal{C}$ the monoid $\text{End}_{\mathcal{P}^{(k)}(\mathcal{C})}(\mathbf{i})$ is regular.*
- (ii) *For each $k \geq 1$ and for each $\mathbf{i} \in \mathcal{C}$ the monoid $\text{End}_{\mathcal{P}^{(k)}(\mathcal{C})}(\mathbf{i})$ is a retract of the monoid $\text{End}_{\mathcal{P}^{(k+1)}(\mathcal{C})}(\mathbf{i})$.*

Proof. In the case $k = 2$ the statement (i) follows from the following diagram, in which all squares are pullbacks:

$$\begin{array}{ccccccc}
 & & & & & & \mathbf{j} \\
 & & & & & & \uparrow \gamma \\
 & & & & \mathbf{i} & \xleftarrow{\alpha} & \mathbf{1} & \xleftarrow{\beta} & \mathbf{k} \\
 & & & & \uparrow \alpha\beta & & \uparrow & & \uparrow \\
 & & & & \mathbf{k} & \xleftarrow{\gamma} & \mathbf{k} & & \mathbf{k} \\
 & & & & \uparrow \gamma & & \uparrow \beta & & \uparrow \\
 & & & & \mathbf{j} & \xleftarrow{\gamma} & \mathbf{k} & & \mathbf{k} \\
 & & & & \uparrow \gamma & & \uparrow \beta & & \uparrow \\
 \mathbf{i} & \xleftarrow{\alpha} & \mathbf{1} & \xleftarrow{\beta} & \mathbf{k} & = & \mathbf{k} & = & \mathbf{k} & = & \mathbf{k} & = & \mathbf{k} & = & \mathbf{k}
 \end{array}$$

In the general case the argument is the same (but requires more space to draw).

The statement (i) follows immediately from (6.4) and Lemma 16. \square

Applying Theorem 18 to the category \mathcal{C}_1 from Subsection 3.1 we obtain a series of regular monoids for which \mathcal{IS}_n is a retract (and such that each monoid in the series is a retract of the next one). These monoids might be interesting objects to study. Later on in Subsection 6.10 we shall discuss slightly different orthodox generalizations of \mathcal{IS}_n .

6.4. $\mathcal{P}(\mathcal{C})$ does not have to satisfy Condition 1. Here we give an example of a category \mathcal{C} satisfying Condition 1 such that $\mathcal{P}(\mathcal{C})$ does not satisfy this condition. Let the objects of \mathcal{C} be the set \mathbb{N} and all its finite subsets. Set

$$\mathcal{C}(X, Y) = \begin{cases} \text{all injections from } X \text{ to } Y, & X \neq \mathbb{N} \text{ or } Y \neq \mathbb{N} \\ \mathbf{1}_{\mathbb{N}}, & X = Y = \mathbb{N}. \end{cases}$$

One easily checks that \mathcal{C} is a category. Exactly in the same way as in Subsection 3.1 one shows that \mathcal{C} satisfies Condition 1.

Let $A = \{1\}$, $B = \{1, 2\}$ and $f = D(\mathbb{N}, B, B, \text{incl}, \mathbf{1}_B)$. We claim that the solid part of the following digram in $\mathcal{P}(\mathcal{C})$ does not have a pullback (note that incl is a monomorphism in $\mathcal{P}(\mathcal{C})$ by Proposition 15):

$$(6.5) \quad \begin{array}{ccc}
 \mathbb{N} & \xrightarrow{f} & B \\
 \uparrow \alpha & & \uparrow \text{incl} \\
 D & \xrightarrow{g} & A
 \end{array}$$

Assume that this is not the case. Fix some $k \in \{1, 3, 4, \dots\}$. Set $D_k = \{1, 3, \dots, k\}$ and denote by ι_k the natural inclusion $D_k \hookrightarrow \mathbb{N}$. Let $g = g_k = D(D_k, A, A, \text{incl}, 1_A)$, $\alpha = \alpha_k = D(D_k, \mathbb{N}, D_k, 1_D, \iota_k)$. A direct calculation shows that the diagram (6.5) commutes for each $k \in \{1, 3, 4, \dots\}$. If a pullback would exist, one easily checks that for the pullback $D \neq \emptyset, \mathbb{N}$. Further, one shows that, without loss of generality, in the pullback we have $D = D_k$ for some $k \in \{1, 3, 4, \dots\}$ and even $g = g_k$ and $\alpha = \alpha_k$. The pullback condition and the commutativity of our digram for $k+1$ would now imply the existence of a map, $\gamma \in \mathcal{P}(\mathcal{C})(D_{k+1}, D_k)$ such that $\iota_{k+1} = \iota_k \gamma$. A direct computation shows that such γ does not exist. A contradiction. Hence $\mathcal{P}(\mathcal{C})$ does not satisfy Condition 1.

6.5. \mathcal{P}^n as an endofunctor. We would like to define some category on which \mathcal{P} would be an endofunctor. After the previous subsection it is clear that we can not just take the category of all categories, satisfying Condition 1. Hence we impose one more condition, which at first glance looks rather artificial. The naturality of this condition will become clear later on, when we show that it works.

Assume that \mathcal{C} is a category, satisfying Condition 1 and the following condition:

Condition 19. For each $\mathbf{i}, \mathbf{j}, \mathbf{k} \in \mathcal{C}$ and monomorphisms $\alpha \in \mathcal{C}(\mathbf{i}, \mathbf{j})$ and $\beta \in \mathcal{C}(\mathbf{j}, \mathbf{k})$ there exists $\mathbf{1} \in \mathcal{C}$ and monomorphisms $\gamma \in \mathcal{C}(\mathbf{i}, \mathbf{1})$ and $\delta \in \mathcal{C}(\mathbf{1}, \mathbf{k})$ such that:

(a) The solid square on the diagram (6.6) is a pullback:

$$(6.6) \quad \begin{array}{ccc} \mathbf{n} & \xleftarrow{\zeta} & \mathbf{k} \\ \uparrow \eta & \searrow \varphi & \nearrow \delta \\ & \mathbf{1} & \\ \downarrow & \uparrow \gamma & \downarrow \beta \\ \mathbf{m} & \xrightarrow{\xi} & \mathbf{j} \\ & \downarrow \alpha & \end{array}$$

(b) For every $\mathbf{m}, \mathbf{n} \in \mathcal{C}$ and monomorphisms $\xi \in \mathcal{C}(\mathbf{m}, \mathbf{i})$, $\eta \in \mathcal{C}(\mathbf{m}, \mathbf{n})$ and $\zeta \in \mathcal{C}(\mathbf{n}, \mathbf{k})$ such that the outer square on the diagram (6.6) commutes and is a pullback, there exists $\varphi \in \mathcal{C}(\mathbf{n}, \mathbf{1})$ making the whole diagram (6.6) commutative.

It is easy to see that in Condition 19 the map φ , if exists, is automatically unique and a monomorphism. Moreover, the left square of the diagram is a pullback (since the outer one is and the diagram commutes). It is also straightforward to verify that $\mathbf{1}$, γ and δ are defined uniquely up to an isomorphism. We will call the right square of (6.6) the *complement diagram*. As an example, later on in Subsection 6.10 we will show that the category \mathcal{C}_1 from Subsection 3.1 satisfies Condition 19.

Denote by \mathcal{S} the category, whose objects are small categories, satisfying Condition 1 and Condition 19, and whose morphisms are all possible functors, which preserve monomorphisms.

Theorem 20. The functor \mathcal{P} is an endofunctor of the category \mathcal{S} .

Proof. Let $\mathcal{C} \in \mathcal{S}$. Then Proposition 15 guarantees that $\mathcal{P}(\mathcal{C})$ satisfies Condition 19. Let $\mathcal{C}' \in \mathcal{S}$ and $F : \mathcal{C} \rightarrow \mathcal{C}'$ be a functor, which preserves monomorphisms. Then again Proposition 15 guarantees that $\mathcal{P}(F)$ preserves monomorphisms. Hence we need only to check that $\mathcal{P}(\mathcal{C})$ satisfies Condition 1.

Because of Proposition 15 we can identify the monomorphisms in the categories \mathcal{C} and $\mathcal{P}(\mathcal{C})$. Let $x = D(\mathbf{i}, \mathbf{j}, \mathbf{k}, \alpha, f)$ and $\beta \in \mathcal{C}(\mathbf{1}, \mathbf{j})$ be a monomorphism.

Consider the pullback diagram given by the right square of the diagram (6.7):

$$(6.7) \quad \begin{array}{ccccc} i & \xleftarrow{\alpha} & k & \xrightarrow{f} & j \\ \uparrow \beta'' & & \uparrow \beta' & & \uparrow \beta \\ n & \xleftarrow{\alpha'} & m & \xrightarrow{f'} & l \end{array}$$

Applying Condition 19 to the monomorphism α and β' we obtain n , α' and β'' . The upper row of (6.7) is the element x . The lower row of (6.7) is an element from $\mathcal{P}(\mathcal{C})(n, l)$, say y . Since both squares of (6.7) are pullbacks, a direct calculation shows that we have the following commutative diagram in $\mathcal{P}(\mathcal{C})$:

$$(6.8) \quad \begin{array}{ccc} i & \xrightarrow{x} & j \\ \uparrow \beta'' & & \uparrow \beta \\ n & \xrightarrow{y} & l \end{array}$$

We claim that (6.8) is in fact a pullback. Assume that for some $p \in \mathcal{C}$ there is a monomorphism $\delta \in \mathcal{C}(p, i)$ and $z \in \mathcal{P}(\mathcal{C})(p, l)$ such that $\beta z = x\delta$. Suppose that $z = D(p, l, q, \xi, g)$ and consider the following diagram, the solid part of which commutes because of (6.7):

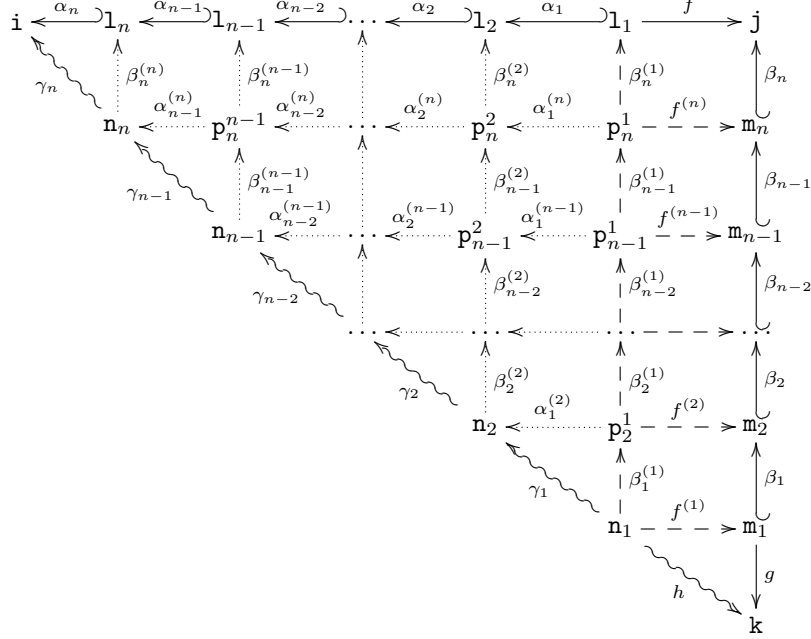
$$(6.9) \quad \begin{array}{ccccccc} & & i & \xleftarrow{\alpha} & k & \xrightarrow{f} & j \\ & \nearrow \delta & \uparrow \beta'' & & \uparrow \beta' & & \uparrow \beta \\ p & \xrightarrow{\varphi} & n & \xleftarrow{\alpha'} & m & \xrightarrow{f'} & l \\ & \searrow \xi & \downarrow \zeta & & \downarrow \eta & & \downarrow g \end{array}$$

Using (2.2) and (2.3) one shows that the condition $\beta z = x\delta$ implies the existence of δ' as on (6.9) such that the part of the diagram (6.9) formed by all solid arrows and δ' commutes and the square $\alpha\delta' = \delta\xi$ is a pullback. Now the fact that the right square is a pullback implies the existence of η as on (6.9) such that the part of (6.9) formed by all solid arrows, δ' and η commutes. Finally, Condition 19 implies now the existence of φ as on (6.9) such that the whole diagram (6.9) commutes.

The commutativity of (6.9) implies $\delta = \beta''\varphi$ and $y\varphi = z$. Hence the diagram (6.8) is a pullback. This completes the proof. \square

6.6. Multiplication for \mathcal{P}^k . Let $\mathcal{C} \in \mathcal{S}$. Then, by Theorem 20 we have that $\mathcal{P}^n(\mathcal{C}) \in \mathcal{S}$ for all $n \geq 0$ (we assume $\mathcal{P}^0 = \text{ID}$). By induction one gets that the morphisms in $\mathcal{P}^n(\mathcal{C})$ are exactly the equivalence classes of the diagrams (6.1) with respect to the equivalence defined on (6.2). However, the multiplication of these elements is quite different from the multiplication in $\mathcal{P}^{(n)}(\mathcal{C})$ described in (6.3). The product in $\mathcal{P}^n(\mathcal{C})$ in terms of the original category \mathcal{C} is described in the following statement:

Proposition 21. Let $D(n, i, j, l_i, \alpha_i, f)$ and $D(n, j, k, m_i, \beta_i, g)$ be two morphisms in $\mathcal{P}^n(\mathcal{C})$. The product $D(n, j, k, m_i, \beta_i, g)D(n, i, j, l_i, \alpha_i, f)$ in $\mathcal{P}^n(\mathcal{C})$ is the element $D(n, i, k, n_i, \gamma_i, h)$ shown on the diagonal of the following commutative diagram (6.10)



where the right column consists of pullbacks and all other small squares are complement diagrams.

Proof. Follows from the construction of pullbacks in the category $\mathcal{P}(\mathcal{C})$ (see the proof of Theorem 20) by induction on n . \square

6.7. Natural transformations. Let $\mathcal{C} \in \mathcal{S}$ and $i, j \in \mathcal{C}$. For $n \geq 1$ and $s \in \{1, 2, \dots, n+1\}$ we define the inclusion $f_n^{(s)} : \mathcal{P}^n(i, j) \rightarrow \mathcal{P}^{n+1}(i, j)$ by mapping the element $D(n, i, j, k_i, \alpha_i, f)$ to the element

$$i \xleftarrow{\alpha_n} l_n \xleftarrow{\alpha_{n-1}} l_{n-1} \xleftarrow{\alpha_{s+1}} k_{s+1} \xleftarrow{\alpha_s} k_s \xleftarrow{\alpha_{s-1}} k_{s-1} \xleftarrow{\alpha_{s-2}} \dots \xleftarrow{\alpha_1} k_1 \xrightarrow{f} j$$

where for convenience we put $k_{n+1} := i$.

Further, for each $n \geq 2$ and each $s, 2 \leq s \leq n$, we define the surjection $t_n^{(s)} : \mathcal{P}^n(i, j) \rightarrow \mathcal{P}^{n-1}(i, j)$ by mapping the element $D(n, i, j, k_i, \alpha_i, f)$ to the element

$$i \xleftarrow{\alpha_n} k_n \xleftarrow{\alpha_{n-1}} \dots \xleftarrow{\alpha_{s+1}} k_{s+1} \xleftarrow{\alpha_s \alpha_{s-1}} k_{s-1} \xleftarrow{\alpha_{s-2}} \dots \xleftarrow{\alpha_1} k_1 \xrightarrow{f} j$$

We have the following statement:

Proposition 22. Let s and n be as above.

- (i) $f_n^{(s)}$ is an injective natural transformation from \mathcal{P}^n to \mathcal{P}^{n+1} .
- (ii) $t_n^{(s)}$ is a surjective natural transformation from \mathcal{P}^n to \mathcal{P}^{n-1} .

- (iii) Both $\mathfrak{t}_{n+1}^{(s)} \mathfrak{f}_n^{(s)}$ and $\mathfrak{t}_{n+1}^{(s+1)} \mathfrak{f}_n^{(s)}$ are the identity transformations.
 (iv) $\mathfrak{t}_{n+1}^{(r)} \mathfrak{f}_n^{(s)} = \mathfrak{f}_{n-1}^{(s)} \mathfrak{t}_n^{(r-1)}$ if $n \geq 2$ and $s+1 < r$.
 (v) $\mathfrak{t}_{n+1}^{(r)} \mathfrak{f}_n^{(s)} = \mathfrak{f}_{n-1}^{(s-1)} \mathfrak{t}_n^{(r)}$ if $n \geq 2$ and $r < s$.

We remark that for the quasi-iteration $\mathcal{P}^{(n)}$ there is no analogue of $\mathfrak{f}_n^{(n+1)}$.

Proof. To prove (i) we have to show that $\mathfrak{f}_n^{(s)}$ behaves well with respect to the composition of morphisms. The latter one is described by the diagram (6.10) (Proposition 21). Applying $\mathfrak{f}_n^{(s)}$ we double one row and one column in (6.10) (inserting the equality signs between the doubled elements). The claim (i) would follow if we would show that the obtained diagram is again of the form (6.10). This reduces (i) to the following facts:

Lemma 23. (a) For any $\mathfrak{i}, \mathfrak{j} \in \mathcal{C}$ and $f \in \mathcal{C}(\mathfrak{i}, \mathfrak{j})$ the diagrams

$$\begin{array}{ccc} \mathfrak{i} & \xrightarrow{f} & \mathfrak{j} \\ \parallel & & \parallel \\ \mathfrak{i} & \xrightarrow{f} & \mathfrak{j} \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathfrak{j} & \xlongequal{\quad} & \mathfrak{j} \\ f \uparrow & & f \uparrow \\ \mathfrak{i} & \xlongequal{\quad} & \mathfrak{i} \end{array}$$

are pullbacks.

(b) For any $\mathfrak{i}, \mathfrak{j} \in \mathcal{C}$ and any monomorphism $\alpha \in \mathcal{C}(\mathfrak{i}, \mathfrak{j})$ the diagrams

$$\begin{array}{ccc} \mathfrak{i} & \xrightarrow{\alpha} & \mathfrak{j} \\ \parallel & & \parallel \\ \mathfrak{i} & \xrightarrow{\alpha} & \mathfrak{j} \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathfrak{j} & \xlongequal{\quad} & \mathfrak{j} \\ \alpha \uparrow & & \alpha \uparrow \\ \mathfrak{i} & \xlongequal{\quad} & \mathfrak{i} \end{array}$$

are complement diagrams to the diagrams

$$\mathfrak{i} \xrightarrow{\alpha} \mathfrak{j} \xlongequal{\quad} \mathfrak{j} \quad \text{and} \quad \mathfrak{i} \xlongequal{\quad} \mathfrak{i} \xrightarrow{\alpha} \mathfrak{j}$$

respectively.

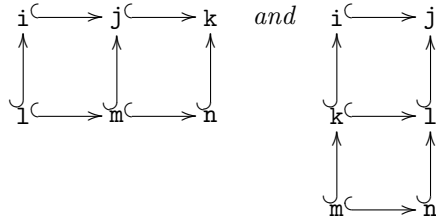
Proof. The whole statement (a) and the statement (b) for the right diagram are obvious. To prove (b) for the left diagram consider the solid part of the commutative diagram

$$\begin{array}{ccc} \mathfrak{k} & \xrightarrow{\beta} & \mathfrak{j} \\ \delta \uparrow & \searrow \gamma \delta^{-1} & \nearrow \alpha \\ \mathfrak{i} & & \mathfrak{j} \\ \parallel & & \parallel \\ \mathfrak{i} & \xrightarrow{\gamma} & \mathfrak{i} \xrightarrow{\alpha} & \mathfrak{j} \end{array}$$

and assume that the outer square is a pullback. Because of the latter assumption and (a) we get that δ is an isomorphism and hence we have the induced map $\gamma \delta^{-1}$ as required by Condition 19. This completes the proof. \square

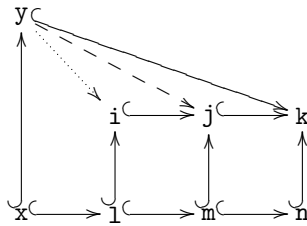
The statement (ii) is more complicated. We again have to show that $\mathfrak{t}_k^{(s)}$ behaves well with respect to the composition of morphisms. Applying $\mathfrak{t}_k^{(s)}$ to (6.10) just forgets one row and one column in the diagram (6.10). The claim (ii) would follow if we would show that the obtained diagram is again of the form (6.10). This reduces (ii) to [Mi, Proposition 7.2] and the following statement:

Lemma 24. *Assume that all small squares on the following diagrams are complement diagrams:*



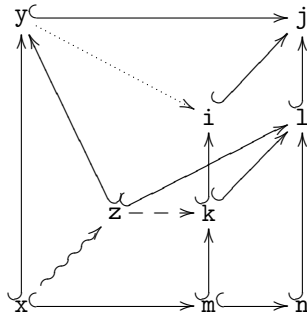
Then the outer rectangles of both diagrams are complement diagrams as well.

Proof. We start from the left diagram (which is easier to deal with). Consider the extended diagram



such that the solid part commutes and the outer square is a pullback. Since the right small square is a complement diagram, we obtain the dashed map, which is automatically a monomorphism. The condition that the outer square is a pullback implies that the square with x, y, j and m is a pullback as well. Now since the middle small square is a complement diagram, we obtain the dotted map, making the whole diagram commutative. This implies that the outer rectangle of our two small squares is a complement diagram.

Now we go on to the right diagram. Consider the following diagram:



Assume that the outer square is a pullback and that the square containing z, y, l and j is a pullback as well. The latter condition implies the existence of the curled map from x to z as indicated. Since the outer square is a pullback, it follows that the square containing x, z, l and n is a pullback as well. Hence we can use the fact that the right bottom small square is a complement diagram and obtain a dashed map (in fact a monomorphism) from z to k as indicated and everything commutes. It is left to observe that the right top small square is a complement diagram, and hence there should exist the dotted map from y to i making the whole diagram commutative. The necessary claim follows. \square

The statements (iii)–(v) are proved by a direct calculation. \square

Altogether we have the following picture:

$$(6.11) \quad \mathcal{P}^0 \xrightarrow{f_0^1} \mathcal{P}^1 \xrightarrow{f_1^1} \mathcal{P}^2 \xrightarrow{f_2^1} \mathcal{P}^3 \xrightarrow{f_3^1} \dots$$

Let \mathcal{A} denote the category, whose objects are \mathcal{P}^n , $n = 0, 1, 2, \dots$, and morphisms are all possible natural transformations of functors.

Let \mathcal{B} denote the category, whose objects are $n \in \mathbb{N}$, and for $m, n \in \mathcal{B}$ we have

$$\mathcal{B}(m, n) = \left\{ \begin{pmatrix} 1 & 2 & \dots & n & \infty \\ a_1 & a_2 & \dots & a_n & a_\infty \end{pmatrix} : \begin{array}{l} a_1 = 1; \quad a_i \leq a_{i+1}; \\ a_\infty = \infty; \quad a_i \in \{1, \dots, m, \infty\} \end{array} \right\}$$

with the obvious multiplication. One shows that any morphisms in the category \mathcal{B} can be written as a composition of the following morphisms:

$$\varphi_n^{(s)} = \begin{pmatrix} 1 & 2 & \dots & s-1 & s & s+1 & s+2 & \dots & n & n+1 & \infty \\ 1 & 2 & \dots & s-1 & s & s & s+1 & \dots & n-1 & n & \infty \end{pmatrix},$$

where $n \in \mathbb{N}$ and $s \in \{1, 2, \dots, n\}$;

$$\varphi_n^{(n+1)} = \begin{pmatrix} 1 & 2 & \dots & n & n+1 & \infty \\ 1 & 2 & \dots & n & \infty & \infty \end{pmatrix}$$

and

$$\tau_n^{(s)} = \begin{pmatrix} 1 & 2 & \dots & s-1 & s & s+1 & \dots & n-1 & n & \infty \\ 1 & 2 & \dots & s-1 & s+1 & s+2 & \dots & n & n+1 & \infty \end{pmatrix},$$

where $n > 1$ and $s \in \{2, \dots, n\}$. Denote by \mathcal{O}_n the monoid of all order-preserving transformations on the chain $\{1, \dots, n, \infty\}$, see for example [Gl], and by \mathcal{O}'_n the submonoid of \mathcal{O}_n consisting of all transformations, which fix the points 1 and ∞ . One easily shows that for $n \in \mathbb{N}$ the monoid $\text{End}_{\mathcal{B}}(n)$ is isomorphic to \mathcal{O}'_n .

Proposition 25. *The assignment $n \mapsto \mathcal{P}^n$, $\varphi_n^{(s)} \mapsto f_n^{(s)}$, and $\tau_n^{(s)} \mapsto t_n^{(s)}$, extends to a faithful functor, $F : \mathcal{B} \rightarrow \mathcal{A}$.*

Proof. A direct calculation shows that F exists. Let $\varphi, \psi \in \mathcal{B}(m, n)$. To claim that F is faithful it is enough to find $\mathcal{C} \in \mathcal{S}$ and $\mathbf{i} \in \mathcal{C}$ such that $F(\varphi)$ and $F(\psi)$ induce different morphisms from $\text{End}_{\mathcal{P}^m(\mathcal{C})}(\mathbf{i})$ to $\text{End}_{\mathcal{P}^n(\mathcal{C})}(\mathbf{i})$. Later on in Subsection 6.10 we will show that the category \mathcal{C}_1 from Subsection 3.1 belongs to \mathcal{S} . The property above is then easily verified by a direct calculation if one takes \mathbf{i} to be a finite set of cardinality at least $\max(m, n) + 1$. \square

As an immediate corollary we obtain:

Corollary 26. *Let $\mathcal{C} \in \mathcal{S}$ and $\mathbf{i} \in \mathcal{C}$. Then the functor F induces an action of the monoid \mathcal{O}'_{n-1} on the monoid $\text{End}_{\mathcal{P}^n(\mathcal{C})}(\mathbf{i})$ by endomorphisms.*

6.8. Connection between $\mathcal{P}^{(n)}$ and \mathcal{P}^n . As we have seen, for $\mathcal{C} \in \mathcal{S}$ the morphism sets in $\mathcal{P}^{(n)}(\mathcal{C})$ and $\mathcal{P}^n(\mathcal{C})$ can be canonically identified. However, the products are rather different. Nevertheless, there is a clear connection between them, which can be described as follows: Let us for the moment denote the product in $\mathcal{P}^{(n)}(\mathcal{C})$ by $*$ and the product in $\mathcal{P}^n(\mathcal{C})$ by \star . We have the following:

Proposition 27. *Let $x = D(n, \mathbf{i}, \mathbf{j}, \mathbf{l}_i, \alpha_i, f)$ and $y = D(n, \mathbf{j}, \mathbf{k}, \mathbf{m}_i, \beta_i, g)$. Set $\mathbf{a} = f_{n-1}^{(1)} \circ \dots \circ f_2^{(1)} \circ f_1^{(1)} \circ t_2^{(2)} \circ \dots \circ t_{n-1}^{(n-1)} \circ t_n^{(n)}$. Then $x * y = x \star \mathbf{a}(y)$.*

Proof. Follows from Proposition 21 and Lemma 23 by a direct calculation. \square

Remark 28. The statement of Proposition 27 reminds of the following fact: if (S, \star) is a semigroup and σ is an idempotent endomorphism of S (retraction), then $(S, *)$, where $x * y := x \star \sigma(y)$, is a semigroup.

6.9. \mathcal{P}^n and orthodox semigroups.

Theorem 29. Let $\mathcal{C} \in \mathcal{S}$ be such that all morphisms in \mathcal{C} are monomorphisms.

- (i) Let $x = D(n, \mathbf{i}, \mathbf{j}, \mathbf{k}_i, \alpha_i, f)$ and $y = D(n, \mathbf{j}, \mathbf{i}, \mathbf{l}_i, \beta_i, g)$. Then x and y form a pair of inverse elements if and only if there is an isomorphism $\gamma : \mathbf{k}_1 \rightarrow \mathbf{l}_1$ such that $\alpha_n \dots \alpha_2 \alpha_1 = g\gamma$ and $f = \beta_n \dots \beta_2 \beta_1 \gamma$.
- (ii) For each $k \geq 0$ and for each $\mathbf{i} \in \mathcal{C}$ the monoid $\text{End}_{\mathcal{P}^k(\mathcal{C})}(\mathbf{i})$ is regular and is a retract of the monoid $\text{End}_{\mathcal{P}^{k+1}(\mathcal{C})}(\mathbf{i})$.
- (iii) The element $D(n, \mathbf{i}, \mathbf{j}, \mathbf{k}_i, \alpha_i, f)$ is an idempotent if and only if $f = \alpha_n \dots \alpha_2 \alpha_1$.
- (iv) For each $k \geq 0$ and for each $\mathbf{i} \in \mathcal{C}$ the monoid $\text{End}_{\mathcal{P}^k(\mathcal{C})}(\mathbf{i})$ is orthodox (i.e. it is regular and its idempotents form a subsemigroup).

Proof. One shows by a direct calculation (using Proposition 21 and Lemma 23) that any pair of elements satisfying the conditions of (i) is inverse to each other. Let x and y be a pair of inverse elements. Consider the retraction \mathbf{a} from Proposition 27. Obviously, $\mathbf{a}(x)$ and $\mathbf{a}(y)$ constitute a pair of inverse elements as well. Now the claim (i) follows from Lemma 3 and Proposition 2 (note that in an inverse semigroup the inverse element is unique).

The statement (ii) now follows from (i) and Proposition 22(iii).

That all elements having the form as in (iii) are idempotents is checked by a direct calculation. Let x be an idempotent. Consider the retraction \mathbf{a} from Proposition 27. Obviously, $\mathbf{a}(x)$ is an idempotent as well. Now the claim (iii) follows from Lemma 4.

From (i) and (iii) it follows that each inverse of an idempotent in the semigroup $\text{End}_{\mathcal{P}^k(\mathcal{C})}(\mathbf{i})$ is an idempotent itself. Hence (iv) follows from (ii) and [How, Theorem 1.1]. \square

6.10. Applications to finite sets.

Proposition 30. $\mathcal{C}_1 \in \mathcal{S}$ (see Subsection 3.1).

Proof. From Subsection 3.1 we know that \mathcal{C}_1 satisfies Condition 1. Hence we have only to check that \mathcal{C}_1 satisfies Condition 19. Let X, Y, Z be finite sets, $\alpha : X \hookrightarrow Y$ and $\beta : Y \hookrightarrow Z$. Set $U = \beta(\alpha(X)) \cup (Z \setminus \beta(Y))$. We have the natural inclusion $\text{incl} : U \hookrightarrow Z$. We also define the map $\gamma : X \rightarrow U$ via $\gamma(x) = \beta(\alpha(x))$. Then γ is obviously an inclusion. A direct calculation shows that the diagram

$$\begin{array}{ccc} U & \xrightarrow{\text{incl}} & Z \\ \uparrow \gamma & & \uparrow \beta \\ X & \xrightarrow{\alpha} & Y \end{array}$$

is a complement diagram. The claim follows. \square

Remark 31. The category $\mathcal{C}_2^{\text{op}}$ from Subsection 3.2 satisfies Condition 1. One can show that it does not satisfy Condition 19.

Proposition 30 allows us to consider the categories $\mathcal{P}^k(\mathcal{C}_1)$ for each $k \in \mathbb{N}$. In particular, for each $n \in \mathbb{N}$ we have an orthodox semigroup

$$\mathcal{RS}(n, k) := \text{End}_{\mathcal{P}^k(\mathcal{C}_1)}(\mathbf{n}),$$

where $\mathbf{n} = \{1, 2, \dots, n\}$. We have $\mathcal{RS}(n, 1) \cong \mathcal{IS}_n$, the classical symmetric inverse semigroup. In what follows we list some basis properties of $\mathcal{RS}(n, k)$ (the proofs are left to the reader).

(I) Each element of $\mathcal{RS}(n, k)$ can be uniquely written in the form

$$(6.12) \quad D(k, A_i, f) : \quad \mathfrak{n} \xleftarrow{\text{incl}} A_k \xleftarrow{\text{incl}} \dots \xleftarrow{\text{incl}} A_2 \xleftarrow{\text{incl}} A_1 \xrightarrow{f} \mathfrak{n},$$

where $A_1 \subset A_2 \subset \dots \subset A_k \subset \mathfrak{n}$, incl denotes the natural inclusion, and $f : A_1 \rightarrow \mathfrak{n}$ is an injection.

- (II) $|\mathcal{RS}(n, k)| = \sum_{i=0}^n \binom{n}{i}^2 i! k^{n-i}$.
- (III) $|E(\mathcal{RS}(n, k))| = (k+1)^n$.
- (IV) $D(k, A_i, f) \mathcal{D} D(k, B_i, g)$ if and only if $D(k, A_i, f) \mathcal{J} D(k, B_i, g)$ if and only if $|A_1| = |B_1|$.
- (V) $D(k, A_i, f) \mathcal{R} D(k, B_i, g)$ if and only if $\text{Im}(f) = \text{Im}(g)$. The \mathcal{R} -class of the element $D(k, A_i, f)$ contains $k^{n-|A_1|}$ idempotents.
- (VI) $D(k, A_i, f) \mathcal{L} D(k, B_i, g)$ if and only if $(A_i) = (B_i)$. Each \mathcal{L} -class of $\mathcal{RS}(n, k)$ contains a unique idempotent.
- (VII) $D(k, A_i, f) \mathcal{H} D(k, B_i, g)$ if and only if $\text{Im}(f) = \text{Im}(g)$ and $(A_i) = (B_i)$.
- (VIII) All maximal subgroups of $\mathcal{RS}(n, k)$ are isomorphic to symmetric groups of rank $\leq n$.
- (IX) $E(\mathcal{RS}(n, k))$ is a Boolean of right singular semigroups, i.e. there is an epimorphism (induced by \mathfrak{a} from Proposition 27), whose image is the Boolean $(2^n, \cap)$ and such that each congruence class of the kernel is a semigroup of right zeros. The kernel of this epimorphism coincides with the minimum semilattice congruence on $E(\mathcal{RS}(n, k))$.
- (X) By (I) and Theorem 29(iii) idempotents in $\mathcal{RS}(n, k)$ are described by flags (A_1, \dots, A_k) of subsets of \mathfrak{n} . In this notation, the multiplication of idempotents is as follows:

$$(B_1, \dots, B_k) \cdot (A_1, \dots, A_k) = ((A_1 \cap B_i) \cup (A_i \setminus A_1))_{i=1}^k.$$

Remark 32. Consider the category \mathcal{C}_{10} from Subsection 3.10. One shows that \mathcal{C}_{10} satisfies Condition 19. One further easily computes that $\text{End}_{\mathcal{P}^k(\mathcal{C}_{10})}(\mathbb{N})$ is a bisimple orthodox monoid for each $k \geq 1$ (it is inverse for $k = 1$).

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